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Holographic Entanglement Entropy and its New Developments

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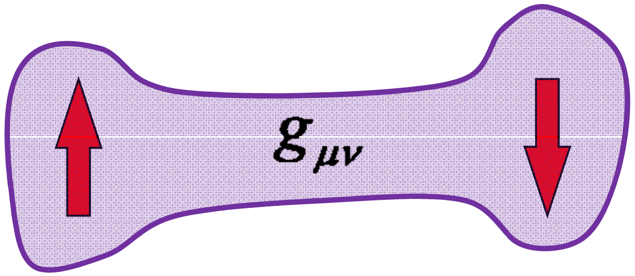

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Tomonori Ugajin (IPMU)

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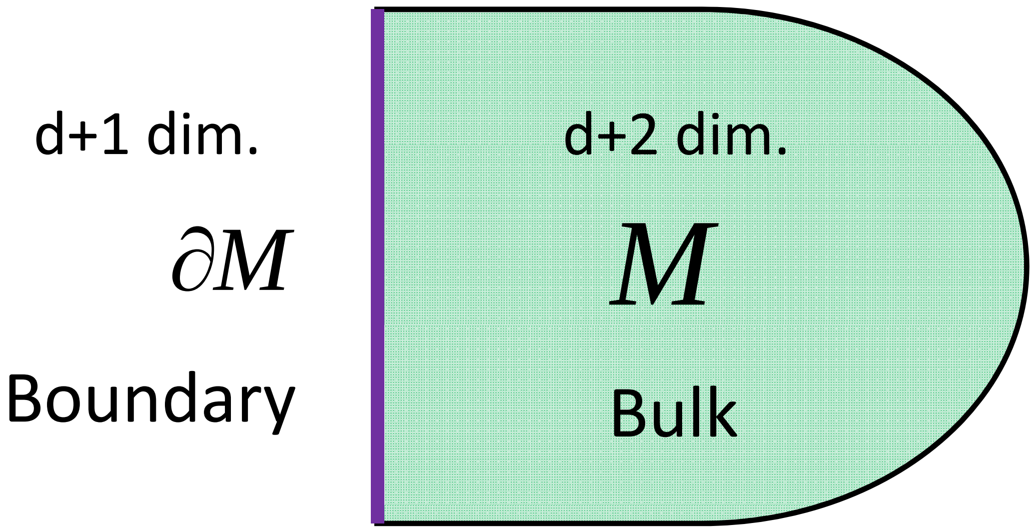
① Introduction

Holography (e.g. AdS/CFT [Maldacena 97])

⇒ Non-perturbative Definition of Quantum Gravity



$$Z_{QM}(\partial M) = Z_{Gravity}(M)$$

To explore the holography in general setups, we need suitable physical quantities. [not only AdS, but flat spaces, de Sitter, etc.]

Stationary BH ⇒ Mass M, Charge Q, Spin J.

(Thermodynamics)

Generic spacetime ⇒ We need much more quantities !

(Non-equilibrium)

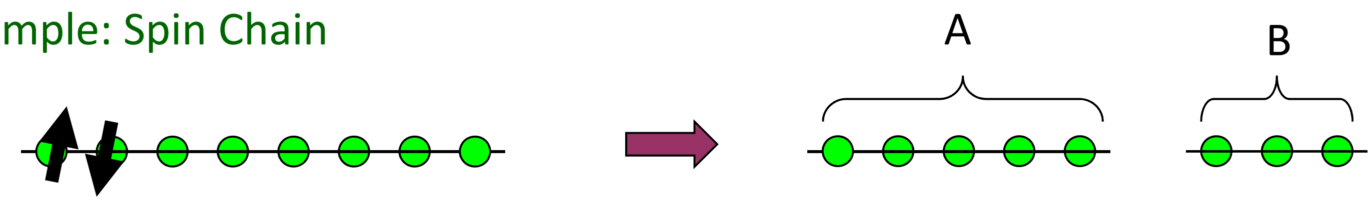
We would like to argue that the entanglement entropy (EE) will be an appropriate quantity.

Definition of Entanglement Entropy

Divide a quantum system into two subsystems A and B.

$$H_{tot} = H_A \otimes H_B \quad .$$

Example: Spin Chain



We define the reduced density matrix ρ_A for A by

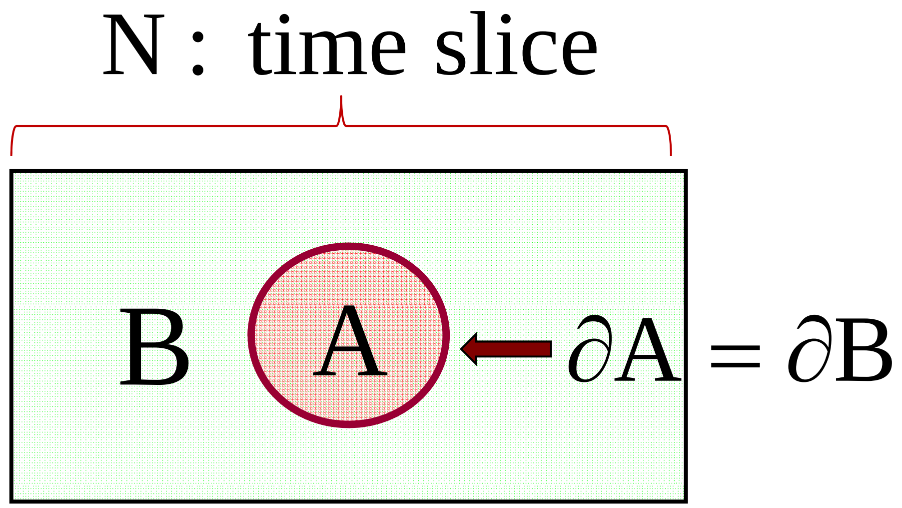
$$\rho_A = \mathrm{Tr}_B \rho_{tot} \quad ,$$

taking trace over the Hilbert space of B .

Now the entanglement entropy S_A is defined by the von-Neumann entropy

$$S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \quad .$$

In QFTs, it is defined geometrically:



Various Applications in other subjects

- **Quantum Information and Quantum Computing**

EE = the amount of quantum information

[see e.g. Nielsen-Chuang's text book 00]

- **Condensed Matter Physics**

EE = Efficiency of a computer simulation (DMRG) [Gaite 03,…]

➡ Divergent at phase transition points !

[G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, 02,…]

i.e. Quantum Critical Points [Sachdev's talk]

➡ A new quantum order parameter !

[Topological entanglement entropy: Kitaev-Preskill 06, Levin-Wen 06]

Basic property: Area law

EE in d+1 dim. QFTs (in the ground states) includes UV div.

[Bombelli-Koul-Lee-Sorkin 86, Srednicki 93]

$$S_A \sim \frac{\mathrm{Area}(\partial A)}{\varepsilon^{d-1}} + (\text{subleading terms}),$$

where ε is a UV cutoff (i.e. lattice spacing).

Similar to the Bekenstein-Hawking formula of black hole entropy

[EE = loop corrections to BH entropy,

Susskind-Uglum 94,...]

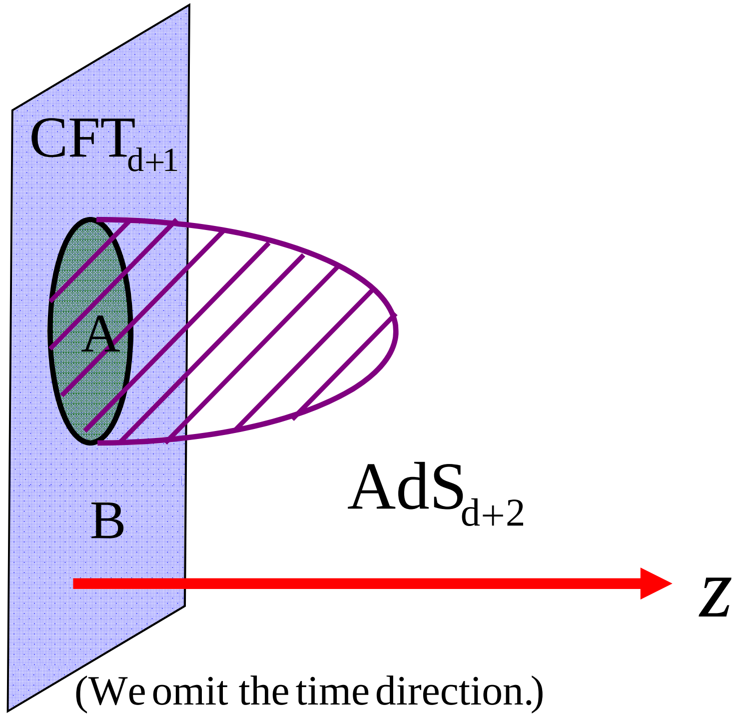
$$S_{BH} = \frac{\mathrm{Area}(\mathrm{horizon})}{4G_N}.$$

② Holographic Entanglement Entropy

(2-1) Holographic Entanglement Entropy Formula

[Ryu-TT 06]

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}$$



γ_A is the minimal area surface (codim.=2) such that

$$\partial A = \partial \gamma_A \text{ and } A \sim \gamma_A \ .$$

Comments

- In the presence of a black hole horizon, the minimal surfaces typically wraps the horizon.

  ⇒ Reduced to the Bekenstein-Hawking entropy, consistently.

- We need to replace minimal surfaces with extremal surfaces in the time-dependent spacetime. [Hubeny-Rangamani-TT 07]

- The area formula assumes the supergravity approximation.

  The holographic formula is modified by higher derivatives.

  [Lovelock: Hung-Myers-Smolkin 11, de Boer-Kulaxizi-Parnachev 11, AdS5 × S5 in IIB String: Ogawa-TT to appear]

- In spite of a heuristic argument [Fursaev, 06] , there has been no complete proof. However, there have been many evidences and no counter examples so far.

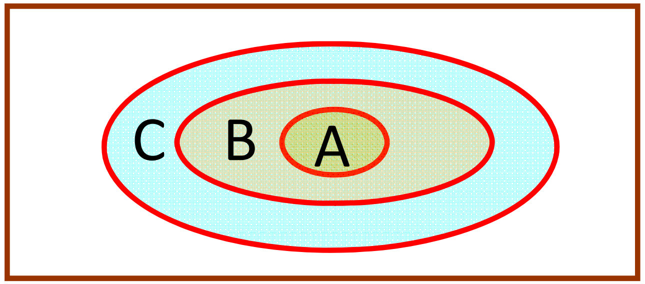
[A Partial List of Evidences]

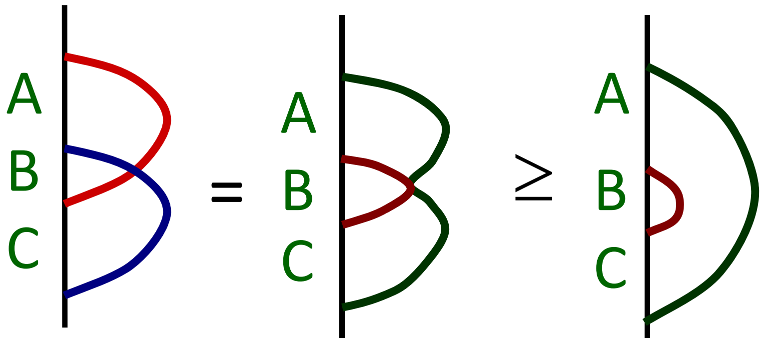
- Area law follows straightforwardly [Ryu-TT 06]
- Agreements with analytical 2d CFT results for AdS3 [Ryu-TT 06]
- Holographic proof of strong subadditivity [Headrick-TT 07]
- Consistency of 2d CFT results for disconnected subsystems [Calabrese-Cardy-Tonni 09] with our holographic formula [Headrick 10]
- Agreement on the coefficient of log term in 4d CFT (~a+c) [Ryu-TT 06, Solodukhin 08,10, Lohmayer-Neuberger-Schwimmer-Theisen 09, Dowker 10, Casini-Huerta, 10, Myers-Sinha 10, Casini-Hueta-Myers 11]

(2-2) Holographic Proof of Strong Subadditivity

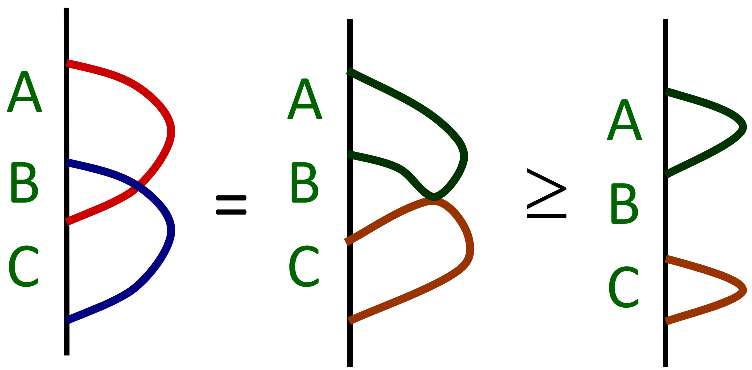
[Headrick-TT 07]

We can easily derive the strong subadditivity, which is known as the most important inequality satisfied by EE. [Lieb-Ruskai 73]





$$\Rightarrow\ S_{A+B}+S_{B+C}\geq S_{A+B+C}+S_{B}$$

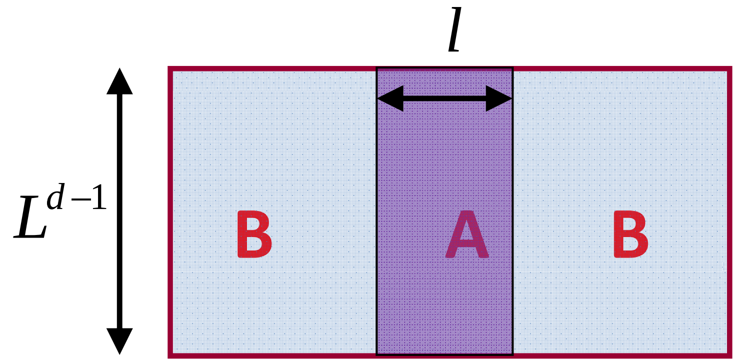


$$\Rightarrow\ S_{A+B}+S_{B+C}\geq S_{A}+S_{C}$$

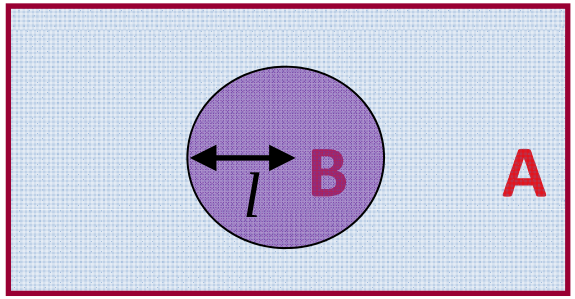
(2-3) Calculations of HEE

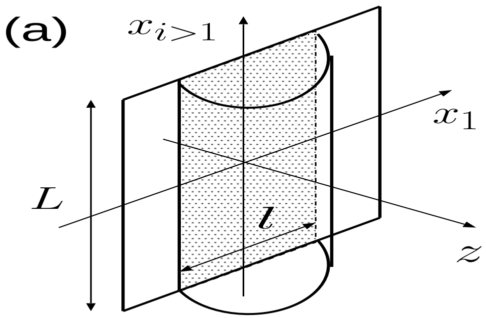
Two analytical examples of the subsystem A:

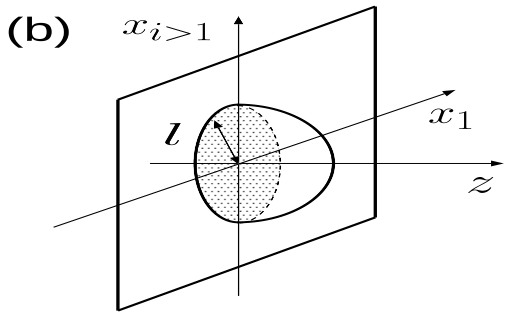
(a) Infinite strip



(b) Circular disk







Entanglement Entropy for (a) Infinite Strip from AdS

$$S_A = \frac{R^d}{2(d-1)G_N^{(d+2)}}\left[\left(\frac{L}{\varepsilon}\right)^{d-1} - C\cdot\left(\frac{L}{l}\right)^{d-1}\right],$$

where $C = 2^{d-1}\pi^{d/2}\left(\Gamma\left(\frac{d+1}{2d}\right)\Big/\Gamma\left(\frac{1}{2d}\right)\right)^d.$

Area law divergence

This term is finite and does not depend on the UV cutoff.

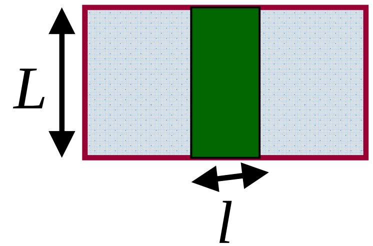
d=1 (i.e. AdS3) case:

$$S_A = \frac{R}{2G_N^{(3)}}\log\frac{l}{\varepsilon} = \frac{c}{3}\log\frac{l}{\varepsilon}.$$

Agrees with 2d CFT results

[Holzhey-Larsen-Wilczek 94 ; Calabrese-Cardy 04]

Basic Example of AdS5/CFT4



$$\mathrm{AdS}_5 \times \mathrm{S}^5 \Leftrightarrow N = 4 \;\; \mathrm{SU(N)} \;\; \mathrm{SYM}$$

$$CFT: \quad S_A^{freeCFT} = K \cdot \frac{N^2L^2}{\varepsilon^2} - 0.087 \cdot \frac{N^2L^2}{l^2}.$$

$$Gravity: \quad S_A^{AdS} = K' \cdot \frac{N^2L^2}{\varepsilon^2} - 0.051 \cdot \frac{N^2L^2}{l^2}.$$

The order one deviation is expected since the AdS result corresponds to the strongly coupled Yang-Mills.

[cf. 4/3 in thermal entropy, Gubser-Klebanov-Peet 96]

Entanglement Entropy for (b) Circular Disk from AdS

[Ryu-TT 06]

$$S_A = \frac{\pi^{d/2} R^d}{2G_N^{(d+2)}\Gamma(d/2)} \left[p_1 \left(\frac{l}{\varepsilon}\right)^{d-1} + p_3 \left(\frac{l}{\varepsilon}\right)^{d-3} + \cdots \right.$$

$$\left. \cdots + \begin{cases} p_{d-1}\left(\frac{l}{\varepsilon}\right) + p_d & (\text{if } d = \text{even}) \\ p_{d-2}\left(\frac{l}{\varepsilon}\right)^2 + q \log\left(\frac{l}{\varepsilon}\right) & (\text{if } d = \text{odd}) \end{cases} \right],$$

where $p_1 = (d-1)^{-1}, p_3 = -(d-2)/[2(d-3)], \ldots .$

$\ldots\ldots \quad q = (-1)^{(d-1)/2}(d-2)!!/(d-1)!! \quad .$

Area law divergence

A universal quantity in odd dimensional CFT

⇒ Satisfy 'C-theorem'

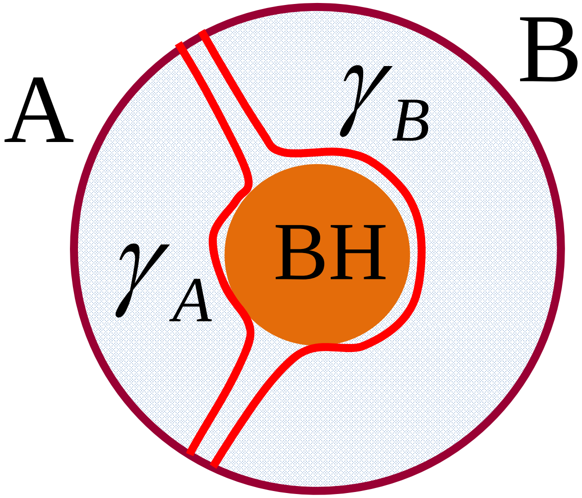
[Myers-Sinha 10]

Conformal Anomaly (~central charge)

2d CFT c/3・log(l/ε)

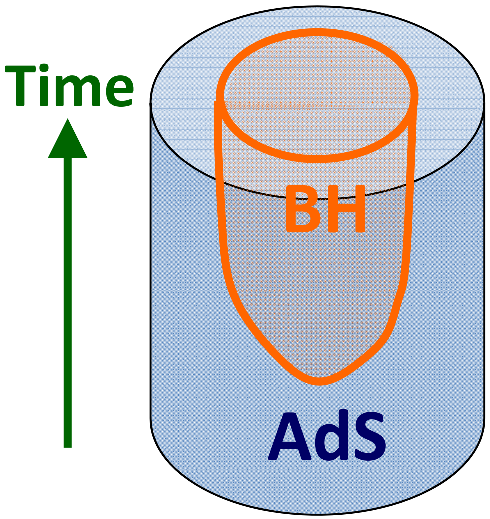
4d CFT -4a・log(l/ε)

(2-4) HEE and AdS BH



$$\text{AdS BH} \Leftrightarrow \text{Finite temp. CFT}$$

$$\rho_{tot} \text{ is not pure } \Leftrightarrow S_A \neq S_B.$$



$$\text{BH formation} \Leftrightarrow \text{Thermalization}$$

$$\rho_{tot} \text{ is pure } i.e. \ S_{tot} = 0,$$

$$\text{but } S_A^{finite} \propto \text{Size of BH.}$$

$$\rightarrow \text{EE} = \text{Coarse-grained entropy}$$

[Arrastia-Aparicio-Lopez 10, Ugajin-TT 10]

③ HEE of Confining Gauge Theories and Higher Derivatives

(3-1) Supergravity Result [Nishioka-TT 06, Klebanov-Kutasov-Murugan 07]

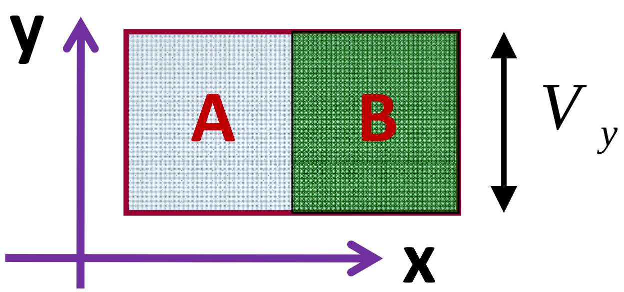
4D N=4 SU(N) SYM on a Scherk-Schwarz circle

⇔ **AdS5 Soliton** × S^5 [Witten 98]

$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + dx^2 + dy^2\right) + \frac{L^2}{r^2}\cdot\frac{dr^2}{1 - r_0^4/r^4} + \frac{r^2}{L^2}\left(1 - r_0^4/r^4\right)d\theta^2.$$

$$\theta \sim \theta + 2\pi R, \qquad \left(R = \frac{L^2}{2r_0}\right).$$

We consider the EE when the subsystem A is just a half space:



Calculation of HEE

$$S_A^{SUGRA} = \frac{\text{Area}}{4G_N^{(5)}} = \frac{2\pi R V_y}{4G_N^{(5)}} \int_{r_0}^{\infty} \frac{r}{L}$$

$$= (\text{area law div.}) - \frac{N^2 V_y}{8R}. \quad \Leftarrow \quad \lambda = \infty$$

Free Field Calculation

After summing over KK modes

$$S_A^{\text{FreeYM}} = (\text{area law div.}) - \frac{N^2 V_y}{12R}. \quad \Leftarrow \quad \lambda = 0$$

⇒ The dependence on λ is non-trivial.

(3-2) Higher Derivative Corrections [Ogawa-TT to appear]

We take into account the R^4 correction in IIB string theory:

$$S_{IIB} = -\frac{1}{16\pi G_N^{(10)}}\int dx^{10}\sqrt{g}\left[R+\frac{\zeta(3)\alpha'^3}{8}W^4+\ldots\right] \quad .$$

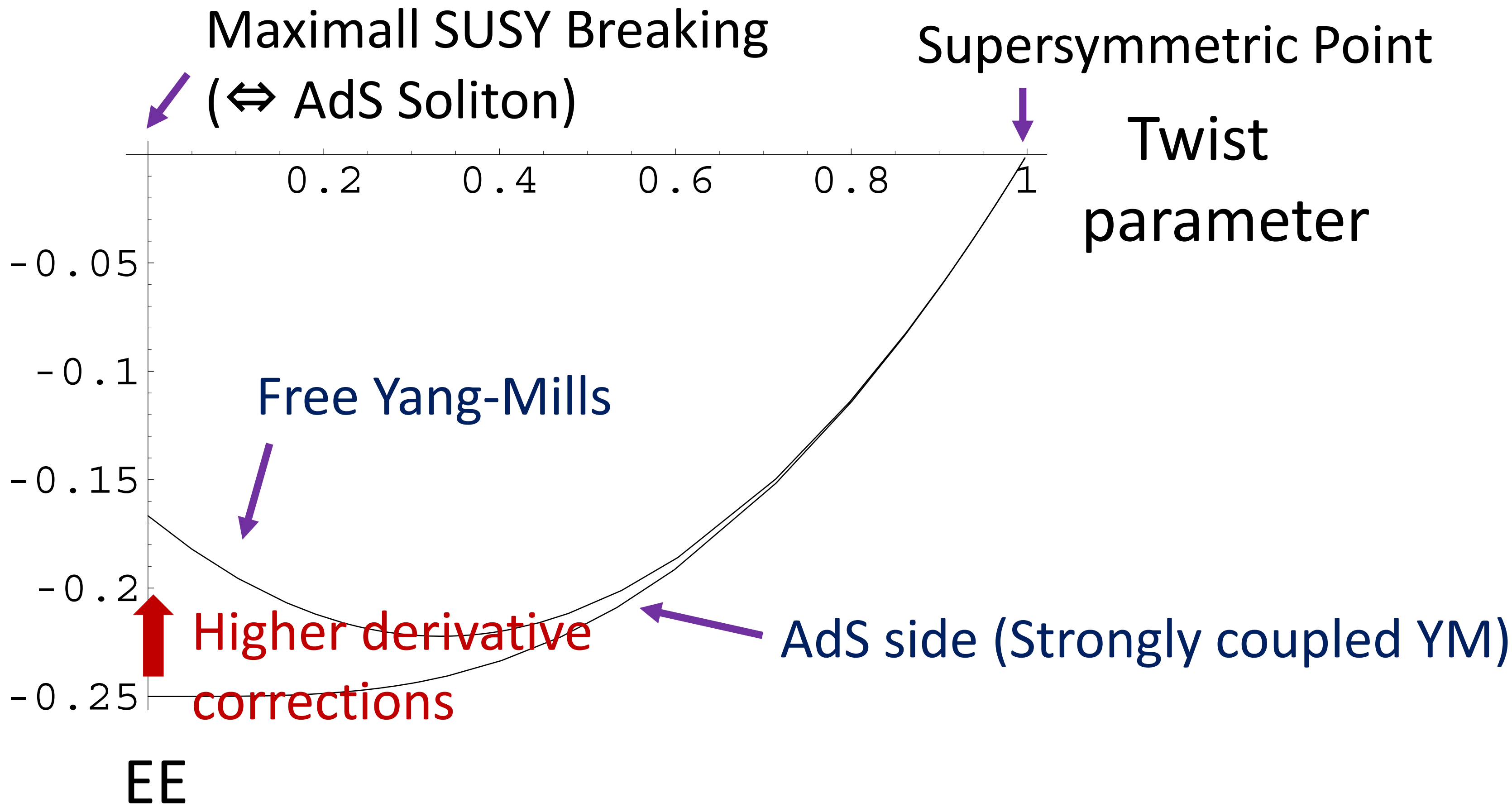
The correction to HEE can be calculated by using the replica trick:

[Cf. thermal entropy: Gubser-Klebanov-Tseytlin 98]

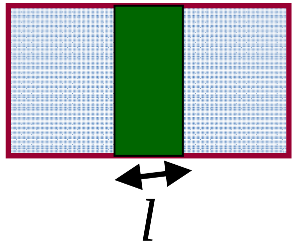
$$S_A = (\text{area law div.}) + \frac{N^2 V_y}{R}\cdot\left(-\frac{1}{8}+\frac{\zeta(3)}{64\sqrt{2}}\lambda^{-\frac{3}{2}}\right).$$

Indeed, HEE increases as
the coupling gets smaller !

N=4 SYM on A Twisted Circle

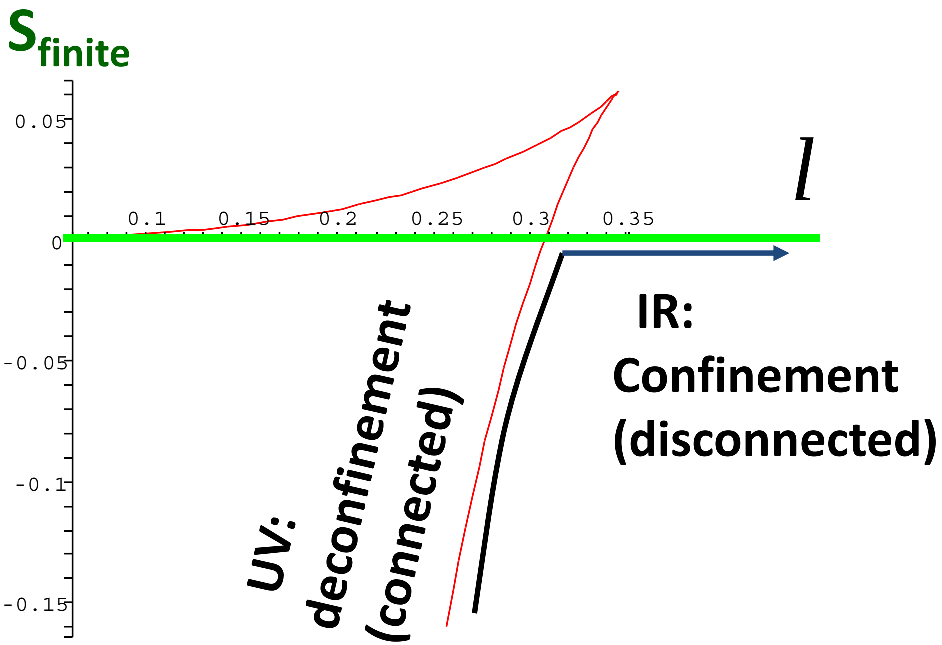


(3-3) Confinement/deconfinement phase transition



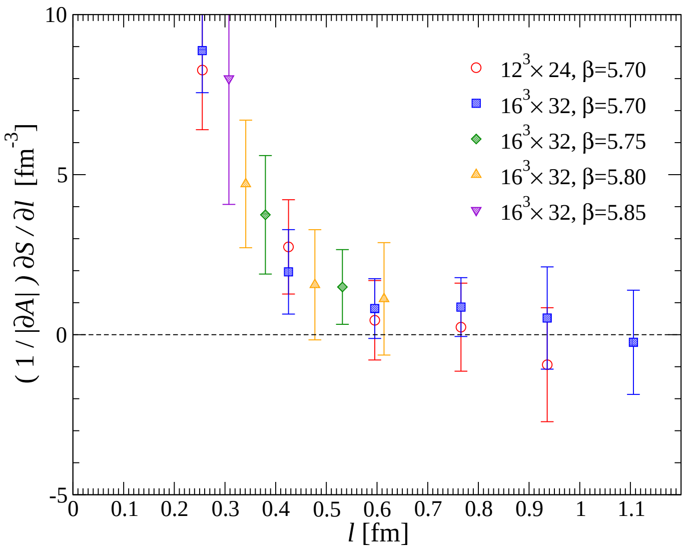
EE can be an order parameter.

HEE of pure SU(N) YM



[Nishioka-TT 06, Klebanov-Kutasov-Murugan 07]

Lattice Result for pure YM



[4D SU(3), Nakagawa-Nakamura-Motoki-Zakharov 09]

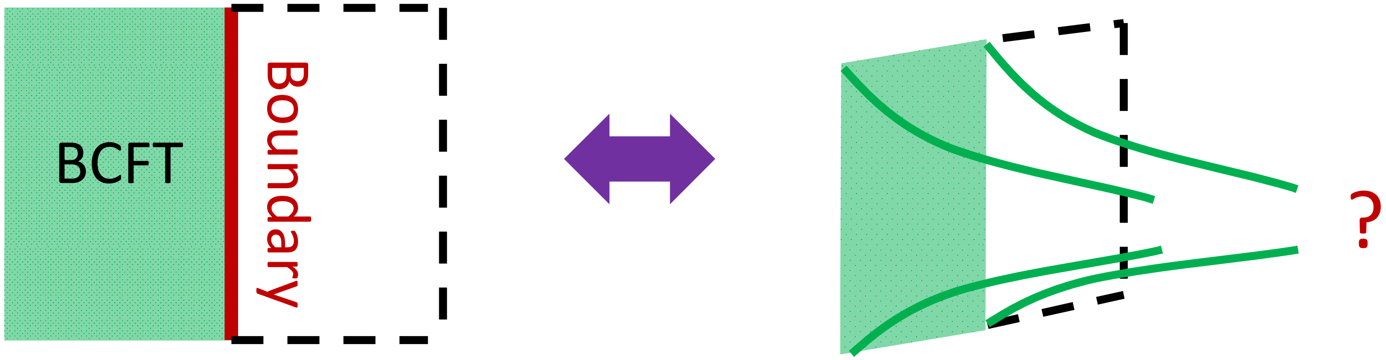
④AdS/BCFT and Quantum Entanglement

(4-1) AdS/BCFT

What is a holographic dual of CFT on a manifold with Boundary (BCFT) ?

CFT_d: SO(d,2) ⇔ AdS_{d+1}

$BCFT_d$: SO(d-1,2) ⇔ AdS_d



[cf. Defect CFT Karch-Randall 00, DeWolfe-Freedman-Ooguri 01,
Janus CFT Bak-Gutperle-Hirano 03, Clark-Freedman-Karch-Schnabl 04]

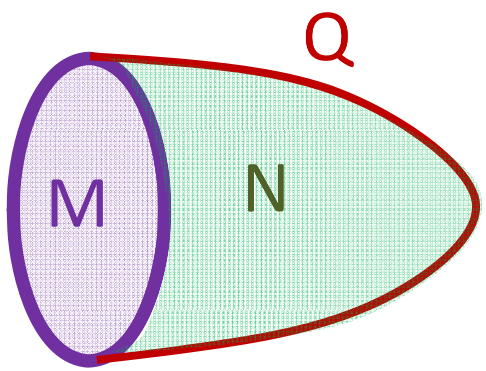
AdS/BCFT Proposal [TT 11 + work in progress with Fujita and Tonni]

In addition to the standard AdS boundary M, we include an extra boundary Q, such that ∂Q=∂M.

$$I_E = -\frac{1}{16\pi G_N}\int_N \sqrt{g}\,(R-2\Lambda - L_{matter}) - \frac{1}{8\pi G_N}\int_Q \sqrt{h}\,(K - L^{Q}_{matter}).$$

EOM at boundary leads to the Neumann b.c. on Q :

$$K_{ab} - K h_{ab} = 8\pi G_N T^{Q}_{ab} \quad .$$



Conformal inv. ⇒ $T^{Q}_{ab} = -T h_{ab}$.

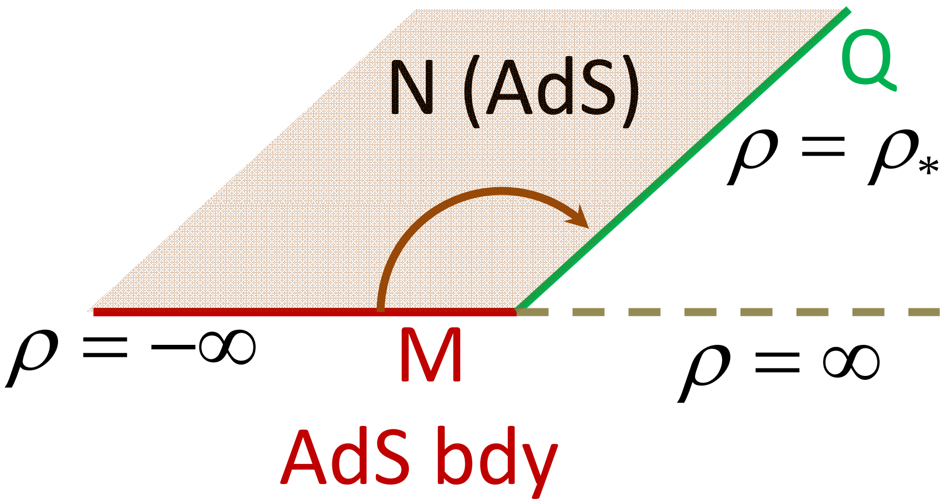
(4-2) Simplest Example

Consider the AdS slice metric:

$$ds^2_{AdS(d+1)} = d\rho^2 + \cosh^2(\rho/R)ds^2_{AdS(d)} \quad .$$

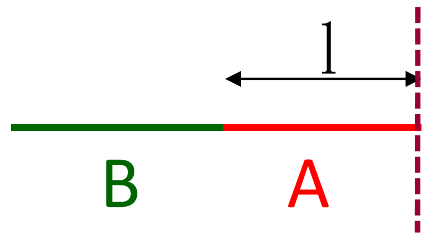
Restricting the values of ρ to $-\infty < \rho < \rho_*$ solves the boundary condition with

$$T = \frac{d-1}{R}\tanh\frac{\rho_*}{R} \quad .$$



(4-3) Holographic Boundary Entropy

The entanglement entropy in 2D CFT
with a boundary looks like
[Holzhey-Larsen-Wilczek 94 ; Calabrese-Cardy 04,
Recent review: Calabrese-Cardy 09, Casini-Huerta 09]



$$S_A = \frac{c}{6}\log\left(\frac{l}{\varepsilon}\right) + \log g \ ,$$

where log g is the **boundary entropy** [Affleck-Ludwig 91].

[Earlier holographic calculations: Yamaguchi 02 (Defect CFT),
Azeyanagi-Karch-Thompson-TT 07 (Non-SUSY Janus),
Chiodaroli-Gutperle-Hung, 10 (SUSY Janus)]

In our setup, HEE can be found as follows

$$S_A = \frac{\text{Length}}{4G_N} = \frac{1}{4G_N}\int_{-\infty}^{\rho_*} d\rho = \frac{c}{6}\log\frac{l}{\varepsilon} + \frac{\rho_*}{4G_N} \quad .$$

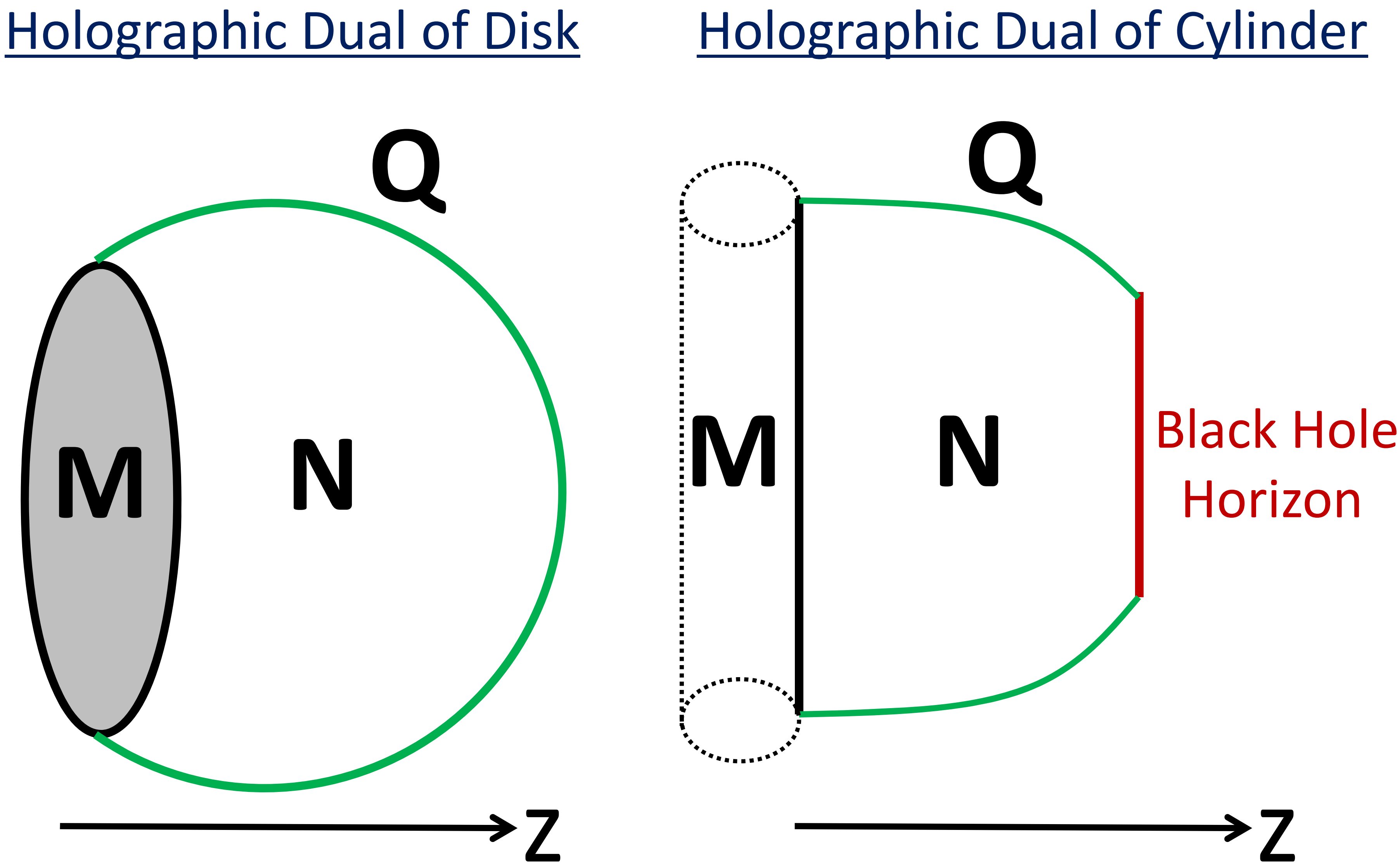
Boundary Entropy

Comment 1. $S_{bdy} = \rho_* / 4G_N$ can be confirmed from the disk cylinder partition function.

$$I_{Disk} = \frac{R}{4G_N}\left(\frac{r^2}{2\varepsilon^2} + \frac{r\sinh(\rho_*/R)}{\varepsilon} + \log\frac{\varepsilon}{r} - \frac{\rho_*}{R} - \frac{1}{2}\right).$$

$$I_{Cylinder} = \frac{\pi}{3} c \cdot l \cdot T_{BH} + \frac{\rho_*}{2G_N}.$$

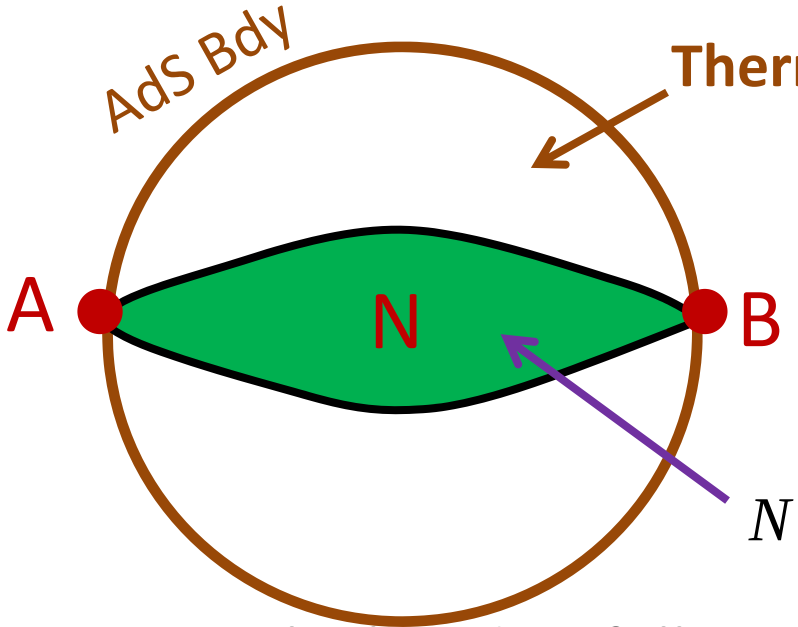
Comment 2. The null energy condition for T^Q leads to a holographic g-theorem.



Holographic Dual of Disk
Q
M
N
Z
Holographic Dual of Cylinder
Q
M
N
Black Hole
Horizon
Z

⑤ Towards Gravity Dual of Lattices [work in progress with Ryu]

(5-1) Holographic Dual of Two Points (or 2 qubits)



$$ds^2 = R^2\left(-\frac{dt^2}{z^2} + \frac{dz^2}{z^2 h(z)} + \frac{h(z)}{z^2}dx^2\right),$$

$$h(z) = 1 - z^2/z_0^2 \ , \quad x \sim x + 2\pi z_0 \ .$$

$$N: \quad |x(z)| \leq z_0 \cdot \arctan\left[\frac{RTz}{z_0\sqrt{h(z) - R^2T^2}}\right]$$

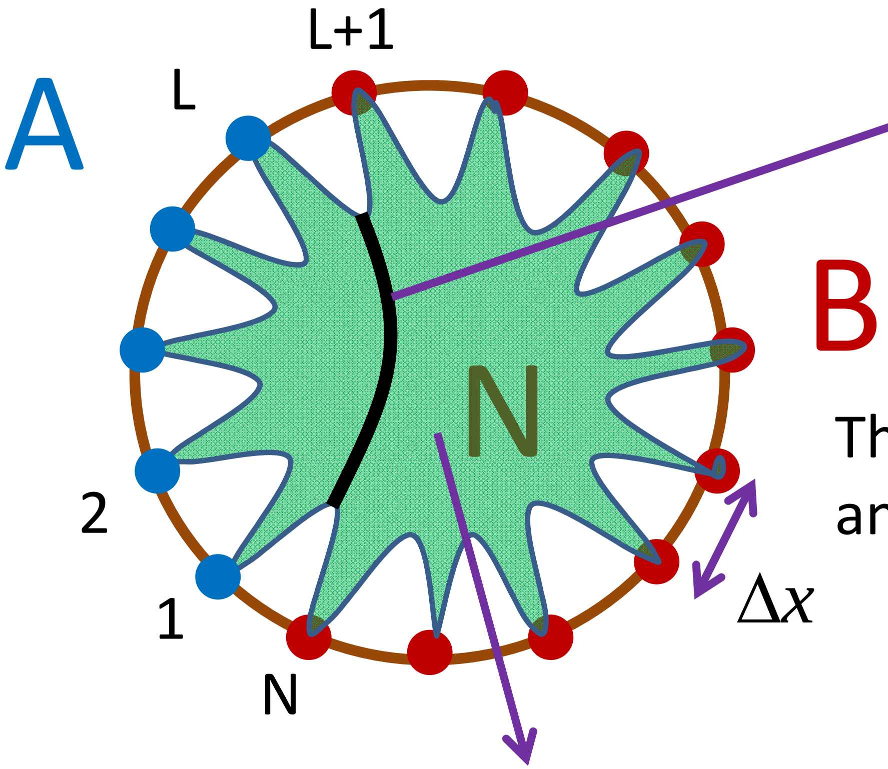
HEE is calculated as follows:

$$S_A = \frac{2R}{4G_N}\int_{z_*}^{z_0}\frac{dz}{z\sqrt{h(z)}} = \frac{2\rho_*}{4G_N} = S_{bdy}$$

➡ A and B are maximally entangled

(5-2) How does Holographic Dual of Lattices look like ?

<u>Holographic dual of many points</u>



Minimal Surface γ_A for A

$$S_A = \frac{|\gamma_A|}{4G_N} \sim \frac{c}{3}\log L.$$

The separation Δx does not have any direct physical meaning in CFT.
But, it does in the AdS gravity.

Emergent AdS space in the IR of a lattice theory
(continuum limit)

This argument looks a bit similar to the calculation framework called MERA (multiscale entanglement renormalization ansatz).

[MERA: Vidal 06; Relation to AdS/CFT: Swingle 09']

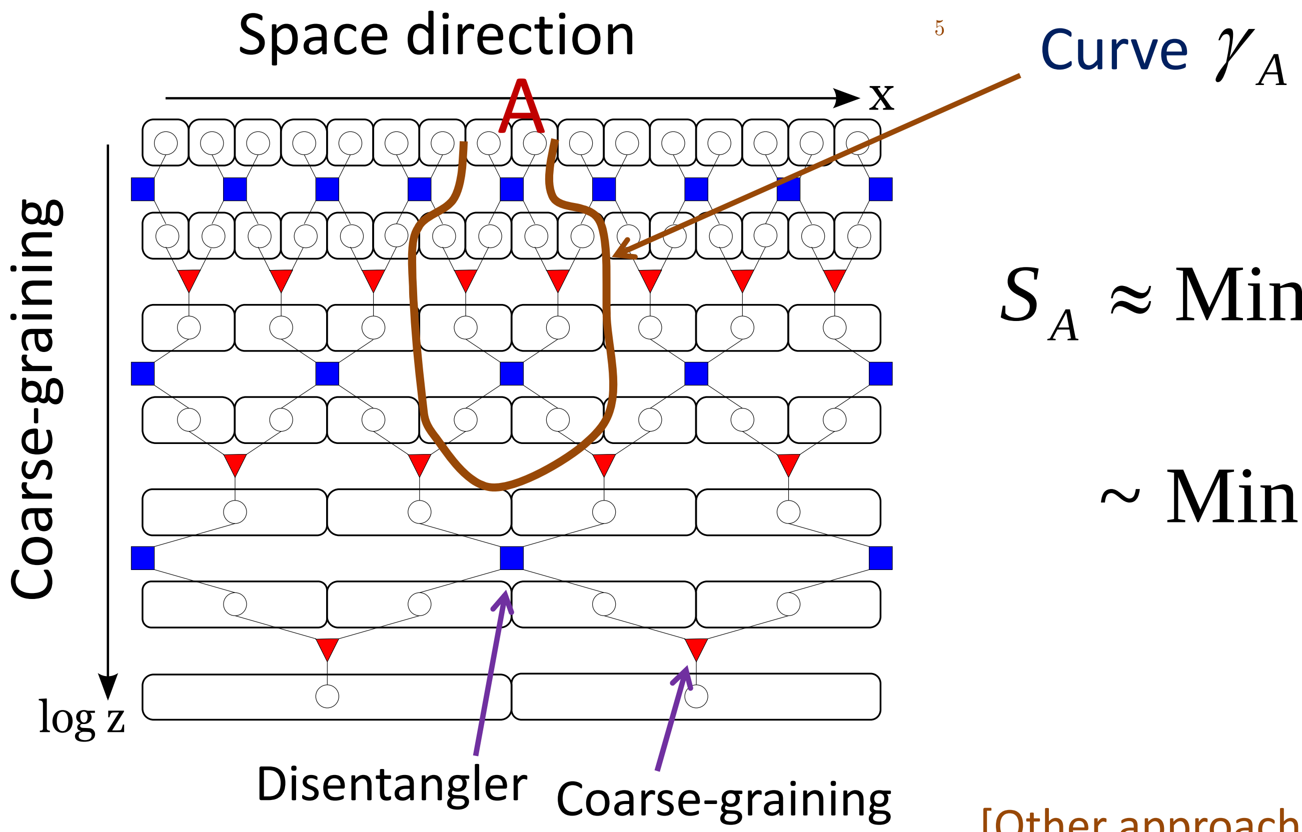


Fig taken from B. Swingle 0905.1317

$$S_A \approx \mathrm{Min}_{\gamma_A}\,[\#\mathrm{Bonds}]$$

$$\sim \mathrm{Min}_{\gamma_A}\left[\frac{|\gamma_A|}{4G_N}\right].$$

[Other approaches to holographic lattices: e.g. Kachru-Karch-Yaida 09, Lee 10]

⑥ Conclusions

- The entanglement entropy (EE) is a useful bridge between gravity and cond-mat physics. [cf. Sachdev's talk]

Gravity	⟺	Entanglement	⟺	Cond-mat.
$g_{\mu\nu}$		$S_A \approx \mathrm{Area}$		systems $\|\Psi\rangle$

- EE can be a universal quantity for holography in general spacetimes.

[e.g. holography in flat space: Li-TT 10

⇒highly entangled and non-local gravity dual

AdS Lorentzian wormholes: Fujita-Hatsuda-TT 11

⇒The EE between two boundaries are actually vanishing.]

- EE is non-zero even for pure states (cf. thermal entropy).
  ⇒ A quantum order parameter at zero temp.

  e.g. Useful for BH formation = Thermalization (quantum quench)

- We proposed a holographic dual of BCFT.
  ⇒ Here again HEE played an important role.
  ⇒ This holography can also be useful in AdS/CMT.

  e.g. Edge states of QHE, Topological Insulators ?
  Any holographic SC localized on boundaries ?
  Non-equilibrium systems with boundaries ?

  :