The Entropy current in Hydrodynamics, Superfluid Hydrodymics and Gravity

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  Based on arXiv:1101.3332 and arXiv:1105.3733
- Other Relevant References:
  arXiv:1004.2707 (Sonner and Withers)
  arXiv:1101.3330 (Herzog, Lisker, Surowka, Yarom)
  arXiv:1104.5245 (Shu Lin)
  arXiv:1106.3576(Neiman and Oz)

Introduction

- In this talk I will present the most general form of the equations of s wave superfluid hydrodynamics at first order in the derivative expansion, subject to the constraints of symmetry and the second law of thermodynamics.
- Although superfluid hydrodynamics is a 60 year old subject, to my knowledge these equations have never correctly been worked out before in full generality
- We needed to obtain these results in order to make sense of certain computations performed within the framework of the so called 'fluid gravity' correspondence of AdS/CFT. I start the talk with a review of this correspondence.

The Fluid-Gravity Map

- 5d negative cosmological constant Einstein Maxwell gravity admits a 5 parameter set of Reisner Nordstrom black brane solutions. The 5 parameters may be taken to be u^μ, T and μ: the 4-velocity, temperature and chemical potential.
- The Goldstone idea may be used to promote these parameters to slowly varying fields. Corresponding to every field configuration of $u^\mu(x)$, $T(x)$ and $\mu(x)$ (subject to an equation of motion see below) there exists a solution to the bulk equations of motion. These solutions can be constructed order by order in $\frac{\partial}{T}$, and have been explicitly constructed to second order in this parameter. The bulk solutions have regular future event horizons.

Constitutive Relations

- These solutions each have an associated boundary stress tensor $T^{\mu\nu}(x)$ and a charge current $J^{\mu}(x)$. $T^{\mu\nu}$ and J^{μ} are determined in terms of the thermodyanmical fields by so called constitutive relations

$$\begin{aligned} T^{\mu\nu} &= (\rho + P)u^{\mu}u^{\nu} + P\eta^{\mu\nu} + \pi^{\mu\nu} \\ J^{\mu} &= qu^{\mu} + J^{\mu}_{diss} \end{aligned} \tag{1}$$

- The pressure $P(T,\mu)$, energy density $\rho(T,\mu)$ and charge density $q(T,\mu)$ are determined by the uniform brane solutions. $\pi^{\mu\nu}$ and J^{μ}_{diss} represent derivative contributions to the constitutive relations. They are determined to m^{th} order by implementation of the fluid-gravity correspondence to the same order.

Equations of Motion and Entropy Current

- Conservation of the boundary stress tensor and current

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= F^{\mu\nu} J_\nu \\ \partial_\mu J^\mu &= cF \wedge F \end{aligned} \tag{2}$$

  yields the equations of fluid dynamics.
- The event horizon, $r_H(x_\mu)$ of the bulk solutions turns out to be a *local* function of thermodynamical fields and their derivatives. The Hodge dual of the pullback of the horizon area form to the boundary yields an entropy current for the fluid flow. Hawking's area increase theorem guarantees that

$$\nabla_\mu J_S^\mu \geq 0$$

The Theory of constitutive relations

- As we have explained, the bulk equations determine the constitutive relations for $\pi^{\mu\nu}$ and J^{μ}_{diss}. Relations depend on the precise form of the bulk equations, and change if, e.g. we couple the Maxwell field to neutral scalars.
- Question: as we vary over all *allowed* bulk couplings do we scan over all symmetry allowed values $\pi^{\mu\nu}$ and J^{μ}_{diss}?
- Answer: No. Generic values for constitutive relations are inconsistent with the positivity of entropy production, as we now explain. The characterization of the most general constitutive relations consistent with non negativity of entropy production is a question addressed by Landau and Lifshitz in their text on Fluid Dynamics. We now repeat their analysis with certain changes.

Listing Fluid Data

- The fluid velocity at any given point breaks the local tangent space $SO(3,1)$ to $SO(3)$. It is useful to label one derivative fluid data by its transformations under $SO(3)$. We start with scalars.
- One derivative scalars
  $u^\mu \partial_\mu T$,
  $u^\mu \partial_\mu \mu$,
  $\partial . u$.
- One derivative scalar EOMs
  $u_\mu \nabla_\nu T^{\mu\nu} = 0$
  $\nabla_\mu J^\mu = 0$
- Onshell independent scalar data
  $S_1 = \partial_\mu u^\mu$

Fluid Data

- Similarly onshell independent vector data
  $V_1 = -P^{\mu\nu}\partial_\nu \frac{\mu}{T} + \frac{F^{\mu\nu}u_\nu}{T}$
  $V_2 = u^\nu \nabla_\nu u^\mu$
  $V_3 = F^{\mu\nu}u_\nu$
- Onshell independent tensor: $\sigma_{\mu\nu}$
- Onshell independent 2 derivative scalar data
  $S_1^T = P^{\mu\nu}\partial_\mu\partial_\nu \left(\frac{\mu}{T}\right)$
  $S_2^T = u^\mu \partial_\mu \partial_\nu u^\nu$
  $S_3^T = \partial_\mu (F^{\mu\nu}u_\nu)$
-

$$P^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$$

$$\sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left(\frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2} - P_{\alpha\beta}\frac{\partial.u}{3}\right)$$

Constitutive Relations Allowed by Symmetry

The most general constitutive relations allowed by symmetry

$$P^{\mu}_{\alpha} P^{\nu}_{\beta} \pi^{\alpha\beta} - \frac{P^{\mu\nu}}{3} P_{\alpha\beta} \pi^{\alpha\beta} = -\eta \sigma^{\mu\nu}$$

$$\frac{(\pi)_{ab} P^{ab}}{3} - \frac{\partial P}{\partial \rho} (u_{\mu} u_{\nu} \pi^{\mu\nu}) + \frac{\partial P}{\partial q} (u_{\mu} J^{\mu}_{diss}) = -\beta \partial_{\alpha} u^{\alpha}$$

$$P^{\mu}_{\alpha} \left(J^{\alpha}_{diss} + \frac{q}{\rho + P} (u_{\nu} \pi^{\alpha\nu}) \right) = \sum_{i=1}^{3} \kappa_i V^{\mu}_i$$

Note that only those parts of $\pi^{\mu\nu}$ and J^{μ}_{diss} that are invariant under field redefinitions of u^{μ}, T and μ are meaningful and may be specified by constitutive relations.

Positivity of Entropy

- We must now impose the requirement of non negativity of entropy production. Landau and Lifshitz proceed as follows. They define a canonical entropy current

$$J^\mu_{cannon} = su^\mu - \frac{\mu}{T} J^\mu_{diss} - \frac{u_\nu \pi^{\mu\nu}}{T}$$

which is asserted, on intuitive grounds, to be the 'correct' entropy current for fluid flows.

- They then use thermodynamics and the equations of motion to demonstrate that

$$\partial_\mu J^\mu_{cannon} = -\partial_\mu \left[\frac{u_\nu}{T}\right] \pi^{\mu\nu} - \partial_\mu \left[\frac{\mu}{T}\right] J^\mu_{diss},$$

and obtain constraints on constitutive coefficients from positivity of the RHS.

- It has become clear in other contexts that the intuition that asserts the 'correct' form of the entropy current is not always infallible. We will not assume it. Instead we simply allow the entropy current to take the most general one derivative form allowed by symmetries
- $$J_S^\mu = J_{canon} + s_1 S_1 u^\mu + \sum_{i=1}^{3} v_i V_i^\mu$$

  We then proceed to compute $\partial_\mu J_S^\mu$.
- The result takes the schematic form

$$\partial_\mu J_S^\mu = \begin{pmatrix} \text{independent two} \\ \text{derivative and curvature data} \end{pmatrix} + \begin{pmatrix} \text{quadratic form in} \\ \text{first order data} \end{pmatrix}$$

- 2 derivative terms in divergence

$$-v_1 S_1^T + (s_1 + v_2) S_2^T + \left(v_3 + \frac{v_1}{T} \right) S_3^T .$$

- Equating to zero
  $v_1 = v_3 = 0$ and $v_2 = -s_1$.
- Resultant entropy current

$$J_S^\mu = J_{canon} + s_1 \left(S_1 u^\mu - V_2^\mu \right)$$

- Apparent one parameter ambiguity

- Move to curved space. Additional term in divergence

$$s_1 R_{\alpha\beta} u^\alpha u^\beta$$

- Thus $s_1 = 0$
- Resultant entropy current

$$J_S^\mu = J_{canon}$$

- Form of entropy current uniquely fixed by requirement of positivity. Agrees with the intuition of Landau and Lifshitz

$$\nabla_\mu J^\mu_S = \frac{S_1}{T}\left[\frac{(\pi)_{ab}P^{ab}}{3} - \frac{\partial P}{\partial \rho}(u_\mu u_\nu \pi^{\mu\nu}) + \frac{\partial P}{\partial q}(u_\mu J^\mu_{diss})\right]$$
$$+ V_{1\,\mu}\left[J^\mu_{diss} + \frac{q}{\rho + P}(u_\nu \pi^{\mu\nu})\right]$$
$$-\frac{\sigma_{\mu\nu}}{T}\left[P^\mu_\alpha P^\nu_\beta \pi^{\alpha\beta} - \frac{P^{\mu\nu}}{3}P_{\alpha\beta}\pi^{\alpha\beta}\right]$$

$$P^\mu_\alpha P^\nu_\beta \pi^{\alpha\beta} - \frac{P^{\mu\nu}}{3}P_{\alpha\beta}\pi^{\alpha\beta} = -\eta\sigma^{\mu\nu}$$
$$P^\mu_\alpha\left(J^\alpha_{diss} + \frac{q}{\rho + P}(u_\nu \pi^{\alpha\nu})\right) = \kappa V^\mu_1$$
$$\frac{(\pi)_{ab}P^{ab}}{3} - \frac{\partial P}{\partial \rho}(u_\mu u_\nu \pi^{\mu\nu}) + \frac{\partial P}{\partial q}(u_\mu J^\mu_{diss}) = -\beta\partial_\alpha u^\alpha$$

Summary- Part 1

- The positivity of entropy production require that 2 out of the 5 possible constitutive parameters vanish. The entropy current is fixed to take the canonical form. Explicit computations in Einstein Maxwell theory bear out both these predictions.
- In order to completely constrain the entropy current we imposed the requirement that the fluid obey the 2nd law even when formulated on a weakly curved background metric.
- Of the 5 first order pieces of data, $\sigma_{\mu\nu}$, $\partial.u$ and V_1^μ are dissipative (result in entropy production). The remaining 2 pieces of data, V_2^μ and V_3^μ are nondissipative to this order.

Hairy black branes

- It was pointed out by Gubser that asymptotically AdS charged black branes are sometimes unstable to the condensation of charged bulk scalars.
- The end point of this instability is a ‘hairy’ black brane; a brane in equilibrium with a charged scalar condensate outside its horizon.
- Under the AdS/CFT duality, a hairy black brane describes a superfluid at rest. The stuff inside the horizon of the black brane is roughly dual the ‘normal’ part of the fluid, while the scalar (Bose) condensate outide the horizon is roughly dual to the superfluid.

- It turns out that hairy black brane solutions admit a natural $d-1$ parameter generalization. It is possible to construct stationary hairy black brane solutions in which which A_i is a nontrivial and reduces to a constant at the boundary , (all fields are functions only of the radial coordinate r in these solutions).
- This solution is dual to a configuration in which the normal fluid is at rest, while the superfluid moves with 4 velocity proportional to $\xi_\mu = \partial_\mu \phi^b - A^b_\mu$ (A^b_μ is the boundary value of the gauge field and ϕ^b is the boundary value of the charged scalar). We can of course produce further solutions by boosting these configurations.
- This is satisfying as superfluids are well known to admit exactly stationary solutions in which the normal fluid moves relative to the superfluid.

- In net, we have an 8 parameter set of stationary solutions labeled by T, u^μ, ξ^μ. T is the temperature of the black brane. u^μ, the boost parameter, may be thought of as the velocity of the normal fluid. A_b^μ encodes the chemical potential of the configuration (via the equation $\mu = A.u$) and the 4 velocity of the superfluid.
- Natural question: what is the value of the stress tensor and charge current in equilibrium configuration labeled by the 8 parameters above?
- In a beautiful paper a year or so ago, Sonner and Withers used abstract (black hole thermodynamics type) reasoning to demonstrate that the answer is given in terms of a single thermodynamical function, $P(T, \mu, \xi)$ by the equations

$$
\begin{aligned}
T^{\mu\nu} &= (\rho_n + P)u^\mu u^\nu + P\eta^{\mu\nu} + \frac{\rho_s}{\xi^2}\xi^\mu\xi^\nu \\
J^\mu &= q_n u^\mu - q_s \frac{\xi^\mu}{\xi} \\
u^\mu \xi_\mu &= \mu
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
\rho_n + P &= q_n\mu + Ts \\
\rho_s &= \mu_s q_s \\
\mu_s &= \xi = \xi_\mu u_s^\mu \\
dP &= sdT + q_s d\mu_s + q_n d\mu_n
\end{aligned}
\tag{4}
$$

Precisely the Landau-Tisza form

Once again we should be able to generate bulk solutions for arbitrarily slowly varying fields $u^\mu(x)$, $T(x)$ and $\xi^\mu(x)$, provided these configurations obey the equations of motion

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= F^{\mu\nu} J_\nu \\ \partial_\mu J^\mu &= cF \wedge F \\ \partial_\mu \xi_\nu - \partial_\nu \xi_\mu &= F_{\mu\nu} \end{aligned} \quad (5)$$

The fluid -gravity map, implemented order by order, generates constitutive relations at the same order. These relations - when combined with constitutive relations - yield equations of motion for the boundary ‘superfluid’

As before, the constitutive relations take the form

$$
\begin{aligned}
T^{\mu\nu} &= (\rho_n + P)u^\mu u^\nu + P\eta^{\mu\nu} + \frac{\rho_s}{\xi^2}\xi^\mu\xi^\nu + \pi^{\mu\nu} \\
J^\mu &= q_n u^\mu - q_s\frac{\xi^\mu}{\xi} + J^\mu_{diss}
\end{aligned}
\tag{6}
$$

- Here the stress tensor is written as a function of the 8 fields $\xi^\mu(x)$, $T(x)$ and $u^\mu(x)$. $\pi^{\mu\nu}$ and J^μ_{diss} represent the contributions of terms that are of atleast first or higher order in the derivatives of fluid fields.
- In order to close those equations we need expressions for $\pi^{\mu\nu}$ and J^μ_{diss}. For every configuration that solves these equations we have a solution of the bulk equations, that has a regular event horizon, and so an entropy current with positive divergence.

- In a corner of parameter regime of a specific model (identified by Herzog) we were able to practically implement the fluid gravity map described abstractly behind to compute $\pi^{\mu\nu}$ and J^{μ}_{diss} explicitly to first order in a derivative expansion
- For technical reasons we worked with a theory of a minimally coupled scalar field of large charge e and $m^2 = -2$. The bulk equations we used were Einstein-Maxwell theory, plus Chern Simons $A \wedge F \wedge F$.
- Our results did not fit within the 13 parameter Clark-Putterman constitutive relations listed in the text book on superfluid hydrodynamics by Putterman. For this reason we were forced to address the question similar to the one answered above: what is the most general form allowed for $\pi^{\mu\nu}$ and J^{μ}_{diss} ?

- The method: imitates the presentation above with some differences. At any given point we have two vectors, $u^\mu(x)$, $\xi^\mu(x)$. Residual symmetry group generically, $SO(3)$. We classify data in representations of $SO(2)$.
- We then proceed to ennumerate onshell indpendent data. For instance, assuming parity invariance we find 7 scalars, 7 vectors and 2 tensors at first order in the derivative expansion.
- We write down the most general constitutive relation allowed by symmetry and then impose non negativity of the divergence of the entropy current. We also impose the so called 'Onsager Reciprocity Relation' to ensure CPT invariance of our results.

- Landau and Lifshitz and Clark and Putterman undertake a similar analysis assuming the entropy current take a canonical form very similar to that described above.
- As in our previous analysis, we proceed without making any assumptions about the form of the entropy current (21 parameters assuming parity, 32 parameters without), but constrain the latter, along with constitutive parameters, via the requirement of positivity, even for superfluid flows in a weakly curved background spacetime.

Results: Parity Even

- If we assume that the superfluid preserves parity, it turns out that (upto a physically inessential ambiguity) the entropy current is uniquely fixed to take the canonical form.
- Dissipative terms are constrained to lie in a 14 parameter family. (Clark and Putterman claimed this was a 13 parameter family, but there was a mistake in their analysis and they missed a parameter). It turns out that the gravity result with zero bulk Chern Simons term does indeed lie within this corrected Clark Putterman family
- In sum the 'Landau Lifshitz' intuition for the structure of the entropy current is impressively borne out in the parity even sector. As a bonus we were able to correct a conceptually minor error in the literature.

Results: Parity Odd

- In this case it turns out that there are 4 additional dissipative coefficients - ignored by Clark and Putterman- even if we assume that the entropy current takes the canonical form.
- More improtantly, however, in this case the entropy current is not uniquely fixed to take the canonical form, but itself has a two parameter ambiguity. It turns out that these two new paramters manifest themselves as two new parameters in the space of allowed dissipative terms, taking the total number of such terms to 20. All this is true both in the presence and in the absence of a $U(1)^3$ anomaly for the current. Our gravitational computations confirm that in specific examples the entropy current does deviate from the canonical form.

Results: Comparison with Gravity

- Although we are still in the process of performing a thorough check, it so far appears that all our gravitational results in the general case (with nonzero bulk Chern Simons term) fit with the 20 parameter prediction described above.
- It is a simple matter to take the non relativistic limit of our 20 parameter set of dissipative terms. No parameters vanish in the process.

- Practitioners of superfluidity are often most interested in the special case in which the normal and superfluid velocities are taken not to deviate much from one another. Formally, one sets these two velocities equal but allows their derivatives to be independent.
- In this special limit Landau-Lifshitz report a 5 parameter set of dissipative terms. Our results agree with theirs in the parity even sector, but include 2 new transport coefficients upon allowing for parity violation
- It does not seem inconceivable that these new terms could show up in a laboratory experiment some day.

- We have derived what we believe to be the most general set of equations for superfluid hydrodynamics at first order in the derivative expansion
- Our results are based on the requirement of the existence of a positive divergence entropy current in an arbitrary curved background spacetime and the Onsager principle
- The superfluid entropy current is forced to agree with the 'Landau Lifshitz' canonical form assuming parity invariance, but in general deviates from this form (with a 2 parameter ambiguity) in the parity odd case
- Our new framework agrees with the results of explicit gravitational computations, to the extent that we have tested it. It is conceivable that our new constitutive parmeters may have observable consequences somewhere

A crazy conjecture

- Recall that the possible forms of the equations of motion of fluid and superfluid hydrodynamics are significantly constrained by the requirement that the entropy never decreases.
- Speculation: Could the same be true of gravity? Could the space of allowed higher derivative corrections to Einstein's equations be significantly constrained (i.e. cut down in number of parameters) by the same requirement?
- Of course when entropy production is positive in Einstein gravity then small higher derivative results cannot change this. However when entropy production vanishes in Einstein gravity, it seems like a possibility that some higher derivative corrections will push the balance the other way.

Crazy Conjecture

- However we have already seen that there exist nontrivial gravitational configurations in which the entropy production vanishes in Einstein gravity: e.g. gravity duals to fluid flows in which V_2 and V_3 are turned on but all other first derivative data is zero.
- Conjecture: The requirement that entropy remain either constant, or increase in such configurations will yield nontrivial constraints on possible higher derivative corrections to Einstein's equation.
- This suggests the exciting possibility of an infinite number of completely new purely low energy constraints on structure of higher derivative corrections to Einstein gravity whose origin lies in the thermodyanical nature of gravity in configurations with a horizon.