Challenges of β-deformation

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I. β-DEFORMATION

β-deformation is an old subject
in the theory of matrix models and symmetric functions

Dedeking function counts Young diagrams

$$\prod_k (1-q^k)^{-1}$$

McMahon formula counts $3d$ partitions

$$\prod_k (1-q^k)^{-k}$$

$$\xleftarrow{t=q} \quad \prod_{i,j} (1-q^i t^j)^{-1}$$

$SL(N)$ characters (Shur fns) $s_R\{p\}$ $\longrightarrow$ MacDonald polynomials $M_R\{p\}$

eigenfunctions of cut-and-join operators $W(\Delta)$,

$$W(\Delta)s_R = \varphi_R(\Delta)s_R$$

$\longleftrightarrow$ eigenfunctions of Ruijsenaars Hamiltonians

$$W(\Delta) = :\prod_i \mathrm{tr}\left(X\frac{\partial}{\partial X}\right)^{\delta_i}:$$

$$p_k = \mathrm{tr}X^k = kt_k$$

Orthogonal polynomials w.r.t. the measure

$$\oint \prod_{i<j}(x_i - x_j)^2 \prod_i \frac{dx_i}{x_i}$$

$$\longrightarrow \quad \oint \prod_{i<j}(x_i - x_j)^{2\beta} \prod_i \frac{dx_i}{x_i}$$

$$\longrightarrow \quad \oint \prod_{i\neq j} \prod_{k=0}^{\beta-1}(x_i - q^k x_j) \prod_i \frac{dx_i}{x_i}$$

$$\begin{array}{ccccc} & & \text{algebra} & & \text{Calogero} \\ & & \text{(Shur fns)} & & \\ & \swarrow & & \searrow & \\ \text{quantum algebra} & & & & \beta - \text{ensemble} \\ \text{(Hall – Littlewood pols)} & & & & \text{(Jack pols)} \\ & \searrow & & \swarrow & \\ & & \text{MacDonald pols} & & \\ \text{Ruijsenaars} & & q, \quad t = q^{\beta} & & \end{array}$$

$$M_1 = p_1 = \mathrm{Tr}\, X = \sum_i x_i, \qquad M_{11} = \frac{1}{2}(-p_2 + p_1^2) = \sum_{i<j} x_i x_j$$

$$M_2 = \frac{1}{2}\left(-\frac{(q+1/q)(t-1/t)}{qt-1/qt}p_2 + \frac{(q-1/q)(t+1/t)}{qt-1/qt}p_1^2\right)$$

$$\ldots$$

Quantum dimensions

Quantum dimensions $M_R^* = M_R\{p = p^*\}$

$$p_k^* = \frac{A^k - A^{-k}}{t^k - t^{-k}} = \frac{\{A^k\}}{\{t^k\}}, \qquad A = t^N$$

$$M_1^* = \frac{A - 1/A}{t - t/t} \quad \overset{t=q}{\longrightarrow} \quad [N]_q \quad \overset{q=1}{\longrightarrow} \quad N$$

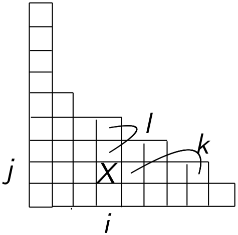
$$M_{11}^* = \frac{\{A/t\}\{A\}}{\{t\}\{t^2\}} \quad \overset{t=q}{\longrightarrow} \quad \frac{[N-1]_q[N]_q}{[2]_q} \quad \overset{q=1}{\longrightarrow} \quad \frac{(N-1)N}{2}$$

$$M_2^* = \frac{\{A\}\{Aq\}}{\{t\}\{qt\}} \quad \overset{t=q}{\longrightarrow} \quad \frac{[N]_q[N+1]_q}{[2]_q} \quad \overset{q=1}{\longrightarrow} \quad \frac{N(N+1)}{2}$$

$$\ldots$$

Hook formula for quantum dimensions:

$$M_R^* = \prod_{(i,j)\in R} \frac{\{Aq^{i-1}/t^{j-1}\}}{\{q^k t^{l+1}\}}$$



Familiar for those who know Nekrasov functions
or topological vertex formulas

$$\{z\} = z - z^{-1}$$

At $\beta \neq 1$ only $M^*_{11\ldots 1}$ are polynomials for $A = t^N$

Today β-deformation is finally in the mainstream:
it *appears* naturally in our theories

Just two examples

AGT

$6d$ CFT compactified on a Riemann surface:
relates smth $2d$ with smth $4d$, e.g.

conformal blocks = LMNS integrals

$$c = (N-1)\left\{1 - N(N+1)\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right)^2\right\}$$

$$g_s = \sqrt{-\epsilon_1\epsilon_2}$$

$$\beta = -\epsilon_2/\epsilon_1 = b^2$$

LMNS integral $= \sum_{R_1,\ldots,R_N}$ Nekrasov functions
Nekrasov functions have typical hook-product form,
similar to MacDonald dimensions

3d AGT

involves $3d$ Chern-Simons theory, e.g.
relates S-duality (modular) transformations with
knot invariants

Wilson average in CS theory = HOMFLY polynomial
of two variables: $q = e^{2\pi i/(k+N)}$ and $a = q^N$

$$\text{HOMFLY}(a|q) \overset{\beta\neq 1}{\longrightarrow} \text{superpolynomial } P(A|q|t)$$

$$P_R[K](A|q|t) = \sum_{Q\vdash b[K]} c_R^Q[K] M_Q^*$$

$$t = q^\beta$$

only quantum dimension M_Q^* depend on $A = t^N$

Coefficients $c_R^Q[K]$ depend on the knot, are rational functions of q, t and for toric knots are described by a simple W-representation

What survives under β-deformation?

Everything related to character calculus:

- Seiberg-Witten equations

$$\left\{\begin{array}{c} a_i = \oint_{A_i} \Omega \\ \\ \frac{\partial \log Z}{\partial a_i} = \oint_{B_i} \Omega \end{array}\right.$$

($\Longrightarrow$ quasiclassical integrability, WDVV equations)

- Virasoro constraints $\longrightarrow$ AMM/EO topological recursion
- W-representations
- AGT relations
- knot invariants

What is lost after β-deformation?

Everything related to KP-integrability:

- $Z = \tau$-function
- determinantal representations
- Harer-Zagier recursion
- Kontsevich matrix models

Nice and natural decompositions:

- AGT could be a Hubbard-Stratanovich duality, but Nekrasov fns have extra poles
- Naive link invariants for $R \neq [1^{|R|}]$ are not superpolynomials

	natural quantities	factorizable constituents
DF integral	Selberg correlators	Nekrasov functions
link invariants	superpolynomials	MacDonald dimensions

II. MATRIX MODELS

Matrix models

- multiple integrals (over eigenvalues)

$$Z = \left(\prod_{i=1}^{N} \int e^{V(x_i)/g_s} dx_i \right) \Delta\{x\}$$

Exact evaluation will one day be possible
in the context of **non-linear algebra** [hep-th/0609022]

Integral discriminants [0911.5278]

$$\iint dxdy\ e^{ax^2+bxy+dy^2} \sim \frac{1}{\sqrt{4ad-b^2}} = D_{2|2}^{-1/2}$$

$$\iint dxdy\ e^{ax^3+bx^2y+cxy^2+dy^3} \sim D_{2|3}^{-1/6}$$

$$D_{2|3} = 27a^2d^2 - b^2c^2 - 18abcd + 4ac^3 + 4b^3d$$

In general ordinary discriminants control
singularities of integral discriminants

Meanwhile – other approaches,
which reveal a lot of hidden structures

• Ward identities (Virasoro constraints; loop equations)
= recursion relations for correlators

$$\left(\sum_k kt_k \frac{\partial}{\partial t_{k+n}} + \sum_{a+b=n} \frac{\partial^2}{\partial t_a \partial t_b}\right) Z = 0$$

– preserved (slightly modified) by β-deformation

• Integrable structure:

as a function of t_k in $V(x) = \sum_k t_k x^k$
Z is a KP/Toda τ-function

$$\frac{\partial^2}{\partial t_1^2} \log Z_N = \frac{Z_{N+1} Z_{N-1}}{Z_N^2}$$

– broken (essentially modified) by the β-deformation

• genus expansion
(t'Hooft coupling $a = Ng_s$ fixed)

$$\Delta = \prod_{i \neq j} (x_i - x_j)^\beta$$

$$\log \Delta \; + \sum_i \frac{1}{g_s} V(x_i) \; \sim \; N^2 \oplus N/g_s$$

$$F = g_s^2 \log Z = \sum_{p=0}^{\infty} g_s^{2p} F_p(a)$$

In perturbation theory F_0 is a sum of planar diagrams and so on.

Many integration contours $\longrightarrow$ many a_I

Spectral curve

- Spectral curve Σ

Σ is defined at the genus-zero level (F_0)

Σ plays prominent role in two places:
resolvents & SW equations

Resolvents, are peculiar generating functions of correlators

$$\rho^{(p|m)}(z_1,\dots,z_m) = \left\langle \prod_{i=1}^{m} \mathrm{Tr}\frac{dz_i}{z_i - X} \right\rangle_p = \sum_{\{k_i\}} \frac{1}{z_i^{k+1}} \left\langle \prod_i \mathrm{Tr} X^{k_i} \right\rangle_p$$

Advantages:

• Resolvents are meromorphic poly-differentials on Σ

• As a consequence of **Virasoro constraints**

they can be recursively reconstructed for a given Σ

+ SW differential $\Omega^{(0)} = \rho^{(0|1)} \sim y(z)dz$ and Bergmann kernel $\rho^{(2|0)}$

(AMM/EO recursion)

Drawback:

• sum over genera diverges, in particular

$$\Omega(z) = \rho^{(\cdot|1)}(z) = \sum_p g_s^{2p} \rho^{(p|1)}(z)$$

can *not* be restored from the AMM/EO recursion

$\rho^{(\cdot|1)}$ as the universal SW differential

$\Omega(z)$ is important:
it is the SW differential for the free energy $F(a) = \sum_p g_s^{2p} F_p(a)$

$$\begin{cases} a_i = \oint_{A_i} \Omega \\ \frac{\partial \log Z}{\partial a_i} = \oint_{B_i} \Omega \end{cases}$$

Always in matrix models and β-ensembles

$$\Omega(z) = \rho^{(\cdot|1)}(z) = \sum_p g_s^{2p} \rho^{(p|1)}(z)$$

(generally believed, but not proved)

Gaussian example: $\Sigma: \quad y(z)^2 = z^2 - 4g_s N$

$$Z_N = \frac{1}{N!}\int (x_i - x_j)^2 e^{-x_i^2/2g_s} dx_i \sim g_s^{N^2/2} \prod_{k=1}^{N-1} k!$$

$$\frac{\partial}{\partial N}\sum_{k=0}^{N-1} f(k) = \sum_k \frac{B_k}{k!}\partial^k f(N)$$

$$\frac{\partial}{\partial N}\log Z_N = N(\log g_s N - 1) + \sum_k \frac{B_{2k}}{k}\frac{1}{N^{2k-1}}$$

$$\Omega(z) = -\frac{y(z)}{2} + \frac{g_s^2}{y(z)^5} + \frac{21 g_s^4 (z^2 + g_s N)}{y(z)^{11}} + \dots$$

$$\oint_A \Omega(z) = N, \qquad \oint_B \Omega(z) = \frac{\partial}{\partial N}\log Z_N$$

General proof $\Longleftarrow$ integrability [1011.5629]

Generalization – theory of DV phases in matrix models

HZ recursion. Alternatives to resolvent

How to define $\rho^{(\cdot|1)}$?

Harer-Zagier recursion $\Longleftarrow$ integrability [1007.4100]

Gaussian model ($V(x) = x^2/2$):

$$\rho(z) = \sum_k \frac{1}{z^{2k+1}} \left\langle \mathrm{Tr} X^{2k} \right\rangle$$

$$\phi(t) = \sum_k \frac{t^{2k}}{(2k-1)!!} \left\langle \mathrm{Tr} X^{2k} \right\rangle$$

$$e(s) = \sum_k \frac{s^{2k}}{(2k)!} \left\langle \mathrm{Tr} X^{2k} \right\rangle$$

$$\left\langle \mathrm{Tr} X^{2k} \right\rangle^{N=1} \sim (2k-1)!! \quad \longrightarrow \quad \left\langle \mathrm{Tr} X^{2k} \right\rangle_0 \sim \frac{(2k-1)!!}{(k+1)!} \ \ \text{(Catalan numbers)}$$

HZ functions for Gaussian model

$$\phi(t|N) = \frac{1}{2t^2}\left(\left(\frac{1+t^2}{1-t^2}\right)^N - 1\right)$$

- $N \longrightarrow \lambda$:

$$\hat{\phi}(t|\lambda) = \sum_{N=0}^{\infty} \phi(t|N)\lambda^N = \frac{\lambda}{\lambda-1}\cdot\frac{1}{1-\lambda-(1+\lambda)t^2}$$

- multi-point correlators:

$$\hat{\phi}_{odd}(t_1,t_2|\lambda) = \frac{\lambda}{(1-\lambda)^{3/2}}\frac{\arctan\frac{t_1t_2\sqrt{1-\lambda}}{\sqrt{1-\lambda+(1+\lambda)(t_1^2+t_2^2)}}}{\sqrt{1-\lambda+(1+\lambda)(t_1^2+t_2^2)}}$$

- other generating functions:

$$\hat{e}(s|\lambda) = \frac{\lambda}{(1-\lambda)^2} e^{\frac{1+\lambda}{1-\lambda}s^2}$$

$$\hat{\rho}(z|\lambda) = \frac{i\lambda}{(1-\lambda)\sqrt{1-\lambda^2}}\ \mathrm{erf}\left(iz\sqrt{\frac{1-\lambda}{1+\lambda}}\right) =$$

$$= \sum_{k=0}^{\infty} \frac{\lambda(1+\lambda)^k}{(1-\lambda)^{k+2}} \frac{(2k-1)!!}{z^{2k+1}}$$

$$\Longrightarrow \quad \rho(z) = \frac{z-y(z)}{2} + \frac{N}{y^5(z)} + \frac{21N(z^2+N)}{y^{11}(z)} + \dots$$

- β-deformation:

$\beta = 2, 1/2$ – 1-point fns through arctan

$\beta = 3$ – diff.eq.

W-representations

Partition functions can be considered as
a result of "evolution", driven by cut-and-join (W) operators
from very simple "initial conditions" [0902.2627]

$$Z\{p\} = e^{g\hat{W}}\tau_0\{p\}$$

If $W \in UGL(\infty)$, then KP/Toda-integrability is preserved

$$\hat{W}_n = \frac{1}{2}\sum_{a,b}\left((a+b+n)p_a p_b \frac{\partial}{\partial p_{a+b+n}} + ab p_{a+b-n}\frac{\partial^2}{\partial p_a \partial p_b}\right)$$

W-representation. Examples

- Hermitian matrix model $\quad Z_N = \int dX e^{\sum_k \frac{p_k}{k} \mathrm{Tr} X^k}$

$$Z_N = e^{\hat{W}_{-2}} e^{N p_0}$$

- Kontsevich model $\quad Z = \int dX e^{\mathrm{Tr}(\frac{1}{3}X^3 - L^2 X)}$, $p_k = \mathrm{Tr} L^{-k}$

$$Z = e^{\hat{W}^K_{-1}} \cdot 1$$

$\hat{W}^K_{-1} = \frac{2}{3} \sum \left(k + \frac{1}{2}\right) \tau_k L^K_{k-1}$ [A.Alexandrov, 1009.4887]

- Hurwitz model [V.Bouchard & M.Marino, 0708.1458]

$$Z = e^{t\hat{W}_0} e^{p_1}$$

- Toric knots and links

$$Z = e^{\frac{n}{m}\hat{W}_0} \prod_{\text{link comps}} \tilde{\chi}_R$$

III. AGT RELATIONS

AGT relations: main points of interest

AGT relation

- Dotsenko-Fateev matrix model
- Hubbard-Stratanovich duality
- Relation to integrable systems
- Bohr-Sommerfeld integrals

Universality classes are labeled by integrable systems

hep-th/9505035

$\mathcal{N}=2$ SYM models	$\overset{\text{AGT}}{\longleftrightarrow}$	$2d$ CFT conformalblocks
$\updownarrow$	dictionary [1995 – 97]	$\updownarrow$
$1d$ integrable systems	$\overset{?}{\longleftrightarrow}$	DF/Penner matrix model

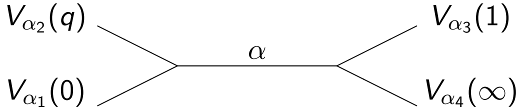
quantization of integrable systems
Shroedinger-like equations (Fourier tr. of Baxter eqs.)
insertions of degenerate states
SW description through BS integrals

$$\Psi(z) = \exp \int^z \Omega, \qquad \Omega = Pdz$$

$$\partial F/\partial a = \oint_B \Omega, \quad a = \oint_A \Omega$$

NS limit $\epsilon_1 \to 0$, $\beta \to \infty$

DF/Penner/Selberg matrix model [ ]



$$\left\langle e^{\alpha_1\phi(0)} e^{\alpha_2\phi(q)} e^{\alpha_3\phi(1)} e^{\alpha_4\phi(\infty)} \prod_{i=1}^{N_1}\int_0^q e^{b\phi(x_i)} \prod_{j=1}^{N_2}\int_0^1 e^{b\phi(y_j)} \right\rangle$$

$$\alpha_1+\alpha_2+bN_1=\alpha$$

$$\alpha+\alpha_3+\alpha_4+bN_2=0$$

$$=\int dx_i\int dy_j\,(x_i-x_{i'})^{2\beta}(y_j-y_{j'})^{2\beta}\underline{(x_i-y_j)^{2\beta}}(x_iy_j)^{2\alpha_1 b}\big((q-x_i)(q-y_j)\big)^{2\alpha_2 b}\big((1-x_i)(1-y_j)\big)^{2\alpha_3 b}$$

$$=\int_{d\mu(x)}\int_{d\mu(y)}\Big(\text{Mixing term}(x|y)\Big)^2$$

Selberg measure

for $\beta = 1$

$$d\mu(x) = \prod_{i<i'} (x_i - x_i')^2 \prod_i x_i^a (1-x_i)^c dx_i$$

is Selberg measure

Natural are Selberg averages of Shur functions,
they are nicely factorized = Nekrasov functions

β and MacDonald deformations:

$$\int_{Jackson} \prod_{k=0}^{\beta-1} \prod_{i \neq i'} (x_i - q^k x_{i'})$$

$$q^\beta = t$$

Averages of Jack and MacDonald polynomials are often *not* factorized
linearly decompose into factorizable quantities (Nekrasov functions)

Pure gauge limit and BGW model

Pure-gauge limit $\longrightarrow$ BGW model

(unitary matrices!) [1011.3481]

Elliptic case (toric conformal blocks)

$\stackrel{?}{\longrightarrow}$ double-cut BGW

[R.Dijkgraaf and C.Vafa, hep-th/0207106]

BGW model is an important building block

in M-theory of matrix models [hep-th/0605171]

AGT as HS duality [1012.3137]

$$\approx \int_{d\mu(x)} \int_{d\mu(y)} \exp\left(2\beta \sum_{i,j} \log(1 - x_i y_j)\right) =$$

$$= \int_{d\mu(x)} \int_{d\mu(y)} \exp\left(\underline{2}\beta \sum_{k} p_k \bar{p}_k / k\right)$$

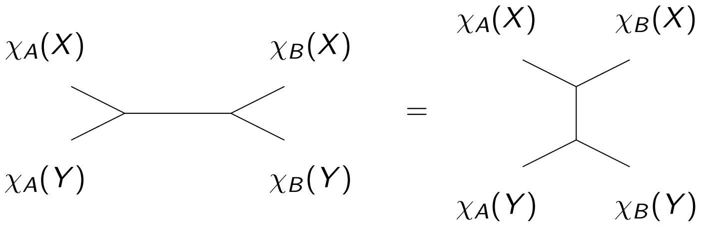
$$= \int_{d\mu(x)} \int_{d\mu(y)} \left(\sum_{A} \chi_A(X)\chi_A(Y)\right)\left(\sum_{B} \chi_B(X)\chi_B(Y)\right)$$

$$= \sum_{A,B} \left(\int_{d\mu(x)} \chi_A(X)\chi_B(X)\right)\left(\int_{d\mu(y)} \chi_A(Y)\chi_B(Y)\right)$$

$p_k = \mathrm{Tr} X^k, \quad \bar{p}_k = \mathrm{Tr}\, Y^k$ [*H.Itoyama & T.Oota* 1003.2929]

$$\exp \sum_{k} \frac{[\beta]_{q^k} p_k \bar{p}_k}{k} = \sum_{A} \frac{C_A}{C_{A'}} M_A(X) M_A(Y)$$

AGT as Hubbard-Stratanovich duality [1012.2137]



$$\sum_{X,Y}\left(\sum_A \chi_A(X)\chi_A(Y)\right)\left(\sum_B \chi_B(X)\chi_B(Y)\right) =$$

$$= \sum_{A,B}\left(\sum_X \chi_A(X)\chi_B(X)\right)\left(\sum_Y \chi_A(Y)\chi_B(Y)\right)$$

$$\text{Conformal block} \;=\; \sum_{A,B} N_{A,B}$$

Decomposition problem for $\beta \neq 1$

$$\int_{d\mu(X)} \chi_A(X)\chi_B(X) \int_{d\mu(Y)} \chi_A(Y)\chi_B(Y) \stackrel{?}{=} N_{A,B}$$

TRUE for $\beta = 1$
NOT so simple for $\beta \neq 1$

$$< \chi_{[1]}\ \chi_\bullet >< \chi_{[1]}\ \chi_\bullet > + < \chi_\bullet\ \chi_{[1]} >< \chi_\bullet\ \chi_{[1]} > =$$
$$= \frac{1}{(z-\epsilon)}\frac{1}{(z+\epsilon)} + \frac{1}{(z+\epsilon)}\frac{1}{(z-\epsilon)} =$$
$$= \frac{2}{z^2-\epsilon^2} = \frac{1}{z(z-\epsilon)} + \frac{1}{z(z+\epsilon)} = N_{[1],\bullet} + N_{\bullet,[1]}$$

For $\epsilon \neq 0$ ($\beta \neq 1$) particular Nekrasov functions
have extra zeroes (at $z = 0$), not present in Kac determinant

Decomposition problem

Instead Nekrasov functions are nicely factorized,
while Selberg correlators for $\beta \neq 1$ are not:

$$< \chi_{[3]}\ \chi_{\bullet} >_{BGW} \sim\ z^2 - (5\epsilon_1 + 8\epsilon_2)z + 6\epsilon_1^2 + 23\epsilon_1\epsilon_2 + 19\epsilon_2^2$$

$$\stackrel{\epsilon_2=-\epsilon_1}{\longrightarrow}\ z^2 + 3\epsilon_1 z + 2\epsilon_1^2 = (z+\epsilon_1)(z+2\epsilon_1)$$

Decomposition problem

Natural quantities, e.g. Selberg correlators
(involved into duality relations)
are linear combinations of
the nicely factorized functions (Nekrasov functions),
which possess extra singularities

Similar is the situation with knot invariants:
superpolynomials for unknots (natural quantitites)
are linear combinations of
MacDonald dimensions (nicely factorized quantities)

IV. KNOTS

Knot theory

Not so much about knots

rather about averages of characters

knot $\longrightarrow$ Wilson average $K = \left\langle \mathrm{Pexp} \oint_{\mathrm{knot}} \mathcal{A} \right\rangle_{CS}$

$$\longrightarrow \; K\{p|\,\mathrm{knot}\} = \sum_R K_R(\mathrm{knot})\chi_R\{p\} \;\longleftrightarrow\; \tau\{p|G\}$$

G – point of the universal moduli space

(universal Grassmannian)

different matrix models – different G

different knots – different G?

modification of τ

A simple example of integrable knot invariants

"Special" polynomials

$$S_R(A) = \Big(S_{[1]}(A)\Big)^{|R|}$$

are obtained from HOMFLY at $q = 1$

Coefficients are Catalan-like numbers,
counting the numbers of certain paths on $2d$ lattices

Satisfy Plücker relations and thus provide KP τ-functions

$$\tau\{p\} = \sum_R S_R(A)\chi_R\{p\}$$

Hierarchy of knot invariants for the $SL(N)$ family

For a given knot K and representation (Young diagram) R

$$
\begin{array}{ccc}
 & \text{Superpolynomial } P_R(A|q|t) & \\
 \swarrow t\approx q & & \searrow A\approx 1 \\
CS \longrightarrow \boxed{\text{HOMFLY } H_R(A|q)} & & \text{Heegard} - \text{Floer } HF_R(q|t) \\
q=1\swarrow & N=2\searrow N=0 & \swarrow t\approx q \\
\text{Special } S_R(A) & \text{Jones } J_R(q) & \text{Alexander } \mathcal{A}_R(q)
\end{array}
$$

$$A = t^N = q^{\beta N}$$

$q = \exp \frac{2\pi i}{k+N}$, $\quad A \sim exp(\text{t'Hooft coupling}),$ finite in the loop expansion

Avatars of the knots

$$\text{knot} \longrightarrow \begin{cases} \text{point of the universal Grassmannian (a dream?)} \\ \\ \text{vector in the Hilbert space,} \\ \text{where modular operators } S \text{ and } T \text{ are acting} \\ \\ \text{vector in the space of characters} \\ \text{(quantum or Macdonald dimensions)} \end{cases}$$

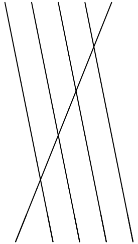
Braid representation of knot average

In the gauge $A_0 = 0$
knot invariants are described in terms of knot diagrams

Can be represented as a braid
Element of a braid group is a product of quantum R-matrices
(some generalization after the β-deformation)
K = "trace" of an element a braid group

Toric knots and links are made from a special braid element

$\mathcal{R}_5$:



Toric links and knots $T[m.n]$:

$$H_R^{[m,n]} = \mathrm{Tr}\left(\mathcal{R}_m\right)^n$$

$$\mathrm{Tr}_Q\ I^{\otimes m} = \mathrm{tr}_Q\ q^{\rho} = \sum_{\vec\alpha\in Q} q^{\vec\rho\vec\alpha} = \chi_Q^*$$

$=$ quantum dimension of representation Q

$$R_1 \otimes \ldots \otimes R_m \ = \ \oplus_{Q\ \vdash\ (|R_1|+\ldots+|R_m|)}\ c_R^Q\ \cdot Q$$

Q – eigenspaces of $\mathcal{R}_m$ with the eigenvalues λ_Q.

$$H_R^{[m,n]}(A|q) \ = \ \sum_Q c_R^Q \lambda_Q^n \chi_Q^* \ = \ e^{n\hat W} \sum_Q c_R^Q \chi_Q\{p\}\Big|_{p=p^*}$$

MacDonald dimensions, repeated

Quantum dimensions $M_R^* = M_R\{p = p^*\}$

$$p_k^* = \frac{A^k - A^{-k}}{t^k - t^{-k}} = \frac{\{A^k\}}{\{t^k\}}, \qquad A = t^N \qquad \{z\} = z - 1/z$$

$$M_1^* = \frac{A - 1/A}{t - t/t} \quad \stackrel{t=q}{\longrightarrow} \quad [N]_q \quad \stackrel{q=1}{\longrightarrow} \quad N$$

$$M_{11}^* = \frac{\{A/t\}\{A\}}{\{t\}\{t^2\}} \quad \stackrel{t=q}{\longrightarrow} \quad \frac{[N-1]_q[N]_q}{[2]_q} \quad \stackrel{q=1}{\longrightarrow} \quad \frac{(N-1)N}{2}$$

$$M_2^* = \frac{\{A\}\{Aq\}}{\{t\}\{qt\}} \quad \stackrel{t=q}{\longrightarrow} \quad \frac{[N]_q[N+1]_q}{[2]_q} \quad \stackrel{q=1}{\longrightarrow} \quad \frac{N(N+1)}{2}$$

$$\ldots$$

HOMFLY case [X.-S.Lin and H.Zheng, math.QA/0601267]:

$$\hat{W} = \frac{1}{m}\hat{W}[2] = \frac{1}{m}\sum_{a,b\geq 1}\left((a+b)p_ap_b\frac{\partial}{\partial p_{a+b}} + abp_{a+b}\frac{\partial^2}{\partial p_a\partial p_b}\right)$$

$$\hat{W}[2]s_Q\{p\} = \varkappa_Q s_Q\{p\}, \qquad \lambda_Q = q^{\varkappa_Q/m}$$

$$s_1\{p\} = p_1, \quad s_2\{p\} = \frac{1}{2}(p_2 + p_1^2), \quad s_{11}\{p\} = \frac{1}{2}(-p_2 + p_1^2), \quad \ldots$$

$$\varkappa_Q = \sum_i q_i(q_i - 2i + 1) = \nu_Q - \nu_{Q'}$$

$$\nu_Q = \sum_i (i-1)q_i$$

For general theory of cut-and-join operators see [0904.4227]

Moreover, "initial conditions" for n-evolution are very simple, e.g.

$$H_1^{[m,n]} = q^{\frac{n}{m}\hat{W}[2]}\ p_m\Big|_{p=p^*}$$

$$H_R^{[m,n]} = q^{\frac{n}{m}\hat{W}[2]}\ s_R\{p_{mk}\}\Big|_{p=p^*}$$

for mutually prime n and m, and

$$H_{R_1\dots R_m}^{[m,mk]} = q^{k\hat{W}[2]}\ s_{R_1}\{p_{mk}\}\dots s_{R_m}\{p_{mk}\}\Big|_{p=p^*}$$

In the last case they simply follow from the fact that $T[m,n]$ for $n=0$ is a set of m unknots.

In the first case for $n=1$ there is a single unknot, i.e. $H_R^{[m,1]} \sim s_R^*$.

Matrix-model representation

$$H_R^{[m,n]}(A|q) \;=\; e^{n\hat{W}} \sum_Q c_R^Q \chi_Q\{p\}\Bigg|_{p=p^*} \;=\; \sum_Q c_R^Q q^{\frac{n}{m}\varkappa_Q} \chi_Q^*$$

Reformulation in terms of Frobenius algebra
(linear space + multiplication + linear form):

$$H_R^{[m,n]}(A=q^N|q) \;=\; \Big\langle s_R[U^m]\Big\rangle = \sum_Q c_R^Q \Big\langle s_Q[U]\Big\rangle$$

$$\Big\langle s_Q[U]\Big\rangle \sim q^{\frac{n}{m}\varkappa_Q} s_Q^*$$

Matrix-model realization of this linear form ($q = e^{\hbar}$):

$$\Big\langle F[U]\Big\rangle = \int du_i e^{u_i^2/\hbar} \sinh\sqrt{\frac{n}{m}}\frac{u_i-u_j}{2} \sinh\sqrt{\frac{m}{n}}\frac{u_i-u_j}{2}\; F\left[\exp\left(\sqrt{\frac{n}{m}}u_i\right)\right]$$

[M.Tierz]; [A.Brini, B.Eynard & M.Marino, 1105.2012]

Deformation from Shur to MacDonald:

$$H_R^{[m,n]}(A|q) = \sum_Q c_R^Q q^{-\frac{n}{m}(\nu_Q - \nu_{Q'})} s_Q^* \quad \longrightarrow$$

$$P_R^{[m,n]}(A|q|t) = \sum_Q c_R^Q \; q^{-\frac{n}{m}\nu_Q} \; t^{\frac{n}{m}\nu_{Q'}} \; M_Q^*$$

split (refined) W-representation
(discrete evolution)

How to choose the coefficients c?

Properties of c_R^Q for toric knots

- They depend on the series $T[m, mk+p]$, $p = 0, 1, \ldots, m-1$
- They satisfy "initial conditions" at $k = 0$: $T[m,p] = T[p,m]$, $p < m$
- They are such, that $P_R^{[m,mk+p]}(A|q|t)$ is a polynomial in all its variables with positive coefficients for all k at once

Initial condition would be sufficient,
if imposed for all values of time-variables p_k

Actually it is imposed only on the subspace $p_k = p_k^* = \frac{A^k - A^{-k}}{t^k - t^{-k}}$,
and this is not sufficient for $|Q| \geq 4$

The third condition should be used

Example of $P_{[1]}^{[m,mk+1]}$

It is tedious, but it works:

$$p_m = \sum_{Q \vdash m} \bar{c}_{[1]}^{Q} M_Q\{p\}$$

$$c_{[1]}^{Q} = \bar{c}_{[1]}^{Q} \cdot \gamma_{[1]}^{Q}$$

$$\gamma^{[2]} = \frac{1+q^2}{1+q^2} = 1, \qquad \gamma^{[11]} = \frac{1+t^2}{1+q^2}$$

$$\gamma^{[3]} = \frac{1+q^2+q^2q^2}{1+q^2+q^2q^2} = 1, \quad \gamma^{[21]} = \frac{1+q^2+q^2t^2}{1+q^2+q^2q^2}, \quad \gamma^{[111]} = \frac{1+t^2+t^2t^2}{1+q^2+q^2q^2}$$

General formula can be easily written down,
also for other series

Verification

- Consistent with all known superpolynomials in all fundamental representations $R = [1^{|R|}]$

- Consistent with HOMFLY – Jones ($N = 2$) – Alexander ($N = 0$) (by definition)

- Consistent with Heegard-Floer polynomials $HF_R(q|t)$

- Consistent with superpolynomials, evaluated by the sums of paths on $2d$ lattices (q, t-Catalan numbers)

- Reproduce $P_{[2]}^{[2,3]}$ of M.Aganagic & Sh.Shakirov,

but does not reproduce Hopf link superpols $P_{[2],[1^s]}^{[2,2]}$ of GIKV and AK (because of the different choice of unknot superpolynomial)

Open problems

- Higher non-fundamental representations $R \neq [1^{|R|}]$

Choice of unknots:

$\frac{Aq-(Aq)^{-1}}{tq-(tq)^{-1}}$ is not a polynomial, even if $A = t^N$

- Link invariants

Do superpolynomials exist at all for toric links?

Weaker polynomiality condition

Weaker positivity condition [Awata & Kanno]

Split W-evolution, starting from modified unknots does not quite reproduce the known answers

- Non-toric knots

Potentially successful example of $5_2 \longrightarrow 10_{139}$

Breakdown of positivity for evolution of 4_1

MANY THANKS FOR YOUR ATTENTION!