Integrability in Quantum Theory, and Applications

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■ Supersymmetric vacua of gauge theories with four supercharges $\Leftrightarrow$ Bethe eigenstates and excitation spectrum of integrable lattice models, Hitchin systems, its limits (quantum many body systems)

• Thermodynamic Bethe ansatz (TBA) type of equations, developed for quantum integrable systems, play the central role

• TBA type equations appear in the study of wall-crossing phenomena in counting of BPS states in N=2 theories

• Correspondence between 4d instanton calculus and two 2d CFT has important consequences, both for CFT and gauge theory

• TBA type equations appear in computing the amplitudes and the expect. values of Wilson and 't Hooft loops for maximal SUSY

• Quantum integrability is central in the study of maximally supersymmetric gauge theories in four dimensions when computing the anomalous dimensions, and in AdS/CFT correspondence

- The spectrum of the equivariant Donaldson theory and its generalizations $\Leftrightarrow$ the spectrum of the quantized SW theory
- Partition functions of closed topological strings $\Leftrightarrow$ the tau-functions of classical integrable hierarchies, and the inclusion of open strings connects to quantum integrability
- Dimer models and their applications to the topological strings on the toric Calabi-Yau manifolds links to the quantum integrability
- Geometric Langlands correspondence, its quantum field theory realization, and the possibility to reach out to number theory
- SLE, random growth and matrix models, emergent geometry

■ Connections and inter-relations with representation theory

- The integrable QFT's in 1+1 dimensions (sine-Gordon, etc.)
- Theory of solitons $\Leftrightarrow$ Classical Inverse Scattering Method, Lax pairs, Spectral curves, etc. and quantization

NS '09: For every quantum integrable system, solved by BA, there is a SUSY gauge theory with 4 supercharges, $Q_+, Q_-, \bar{Q}_+, \bar{Q}_-$ s.t.:

a) exact Bethe eigenstates correspond to SUSY vacua,

b) ring of commuting Hamiltonians $\Leftrightarrow$ (twisted) chiral ring.

- *The effective twisted superpotential* $\Leftrightarrow$ *Yang-Yang function*

$$\boxed{\tilde{W}^{eff}(\sigma) = Y(\lambda)}$$

$$\sigma_i = \lambda_i; \quad i = 1, ..., N; \quad G = U(N)$$

- VEV of chiral ring operators O_k $\Leftrightarrow$ Energies:

$$\boxed{< \lambda|O_k|\lambda >= E_k(\lambda)}$$

$$H_k \Psi(\lambda) = E_k(\lambda)\Psi(\lambda)$$

Vacua/Bethe Equations - critical points of $\tilde{W}^{eff}(\sigma)/Y(\lambda)$ as functions of abelian components of scalar eld σ_i / rapidities λ_i.

What are these quantum integrable systems?

After massive fields (2d) are integrated out chiral ring generators are invariant functions of $\Sigma = \sigma + ...$ on Coulomb branch.

SUSY vacua - there are two options: 1. topological or 2. physical.

1. Topologically twisted (on cylinder) abelianized theory has the action completely determined by $\tilde{W}^{eff}(\sigma)$ of physical theory:

$$S_{top} = \int \left[\frac{\partial \tilde{W}^{eff}(\sigma)}{\partial \sigma_i} F^i(A) + \frac{\partial^2 \tilde{W}^{eff}(\sigma)}{\partial \sigma_i \partial \sigma_j} \lambda^i \wedge \lambda^j \right]$$

$$compare \qquad S_{2d-YM} = \int \left[\sigma_i F^i(A) + \lambda^i \wedge \lambda^j \right]$$

Canonical quantization - momentum conjugate to the monodromy of abelian gauge field $x^i = \int_{S^1} A^i$ is quantized:

$$\frac{1}{2\pi i} \frac{\partial \tilde{W}^{eff}(\sigma)}{\partial \sigma^i} = n_i$$

2. Physical: suppose we have the theory with the effective twisted superpotential $\tilde{W}^{\text{eff}}(\sigma)$ (abelianized).

The target space of the effective sigma model is disconnected, with $\vec{n}$ labeling the connected components (gauge flux quantization) with potential:

$$U_{\vec{n}}(\sigma) = \frac{1}{2}\mathrm{g}^{ij}\left(-2\pi i n_i + \frac{\partial \tilde{W}^{\text{eff}}}{\partial \sigma^i}\right)\left(+2\pi i n_j + \frac{\partial \bar{\tilde{W}}^{\text{eff}}}{\partial \bar{\sigma}^j}\right)$$

Now we need to find the minimum of potential - again:

$$\frac{1}{2\pi i}\frac{\partial \tilde{W}^{\text{eff}}(\sigma)}{\partial \sigma^i} = n_i$$

Or equivalently - SUSY vacua correspond to solution of equation:

$$\exp\left(\frac{\partial \tilde{W}^{\text{eff}}(\sigma)}{\partial \sigma^i}\right) = 1$$

- XXX spin chain - 2d gauge theory

For $SU(2)$, $s = \frac{1}{2}$ spin chain of length L in N-particle sector $\Leftrightarrow$ $U(N)$ 2d $N = 2$ gauge theory with L fundamentals, L anti-funds and 1 adjoint, with twisted masses m_i and complexified θ term; m_i are impurities μ_i, θ - quasi-periodic boundary conditions, ...

- XXZ spin chain - 3d gauge theory on $R^2 \times S^1$

- XYZ spin chain - 4d gauge theory on $R^2 \times T^2$

- Arbitrary spin group, representation, impurities, limiting models

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- NLS, N-particles on S^1, δ-function potential - 2d $\mathcal{N} = 4$ +...

- Periodic Toda - 4d pure $\mathcal{N} = 2$ theory on $R^2 \times R^2_\epsilon$

- Elliptic Calogero-Moser - 4d $\mathcal{N} = 2^*$ theory on $R^2 \times R^2_\epsilon$

Connection to representation theory - 2d $N = 2^*$

Consider 2d pure $N = 4$ gauge theory ($G = U(N)$) broken down to $N = 2$ by the twisted mass (m) term for the adjoint chiral multiplet - $N = 2^*$. Add tree level twisted superpotential:

$$\tilde{W}(\sigma) = \frac{1}{2} tr \sigma^2$$

Vacuum equations:

$$e^{i\sigma_j} = \prod_{i=1}^{N} \frac{\sigma_i - \sigma_j + m}{\sigma_i - \sigma_j + m}$$

For $m = ic$, $c \in R$, this is Bethe equation for NLS quantum theory in N-partical sector.

This is the first example treated in the topological field theory language in MNS '97 and later in GS '06-'07.

This topological theory, YMH theory, computes equaivariant intersection numbers on the moduli space of Higgs bundles introduced by Hitchin:

$$F_{z\bar{z}}(A) = [\Phi_z, \Phi_{\bar{z}}]$$

$$\nabla_z(A)\Phi_{\bar{z}} = 0$$

$$\nabla_{\bar{z}}(A)\Phi_z = 0$$

modulo unitary gauge transformations :

$$A \rightarrow g^{-1}Ag + g^{-1}dg; \quad \Phi \rightarrow g^{-1}\Phi g$$

z - local coordinates on Riemann surface, Φ - adjoint 1-form.

Moduli space of solutions to Hitchin equations - phase space of algebraic integrable system. It is hyperkahler. See later.

NLS in N-particle sector is described by integrable system of N non-relativistic particles on S^1 with δ-function interactions.

$$H_2 = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + c\sum_{1\le i<j\le N}\delta(x_i - x_j)$$

Eigenvectors - spherical vectors in the representation theory of degenerate double affine Hecke algebra.

Latter is connected to the representation theory of $GL(N, Q_p)$ - the wave functions are a limit of Hall-Littlewood polynomials, generalized zonal spherical functions for $GL(N, Q_p)$:

$$\prod_i \frac{1-t}{1-t^{m_i}} \sum_{w\in S_N} (-1)^{l(w)} w(\Lambda_1^{\mu_1}\dots\Lambda_N^{\mu_N}\prod_{i<j}\frac{\Lambda_i - \Lambda_j t}{\Lambda_i - \Lambda_j})$$

$(\mu_1, \dots, \mu_N)$ is a partition of length at most N: $(1^{m_1}, .., r^{m_r}, \dots)$.

NLS wave-functions correspond to analytic continuation with

$$\mu_i = \frac{x_i}{\epsilon}, \quad \Lambda_i = e^{2\pi\epsilon\sigma_i}, \quad t = e^{2\pi i c\epsilon}, \quad \epsilon \to 0 \quad [p \to 1]$$

This is continuous limit of discretized version of H_2.

$GL(N, Q_p)$ zonal spherical functions are Macdonald's $M(q, t)$ for $q = 0, t = p^{-1}$. Eigenfunctions of H_2 discretized (van Diejen, '06).

$M(q, t = q^{\nu})$: relativistic Calogero-Moser-Sutherland (Ruijsenaars '87) $\rightarrow G/G\ WZW$, with Wilson lines (Gorsky, Nekrasov '94).

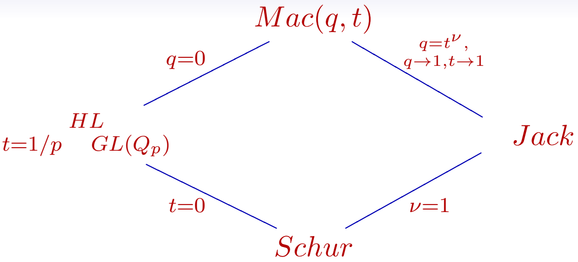
There is another 2d (generalized) $G/G\ WZW$ interpretation which has limit to YMH topological theory for $k \rightarrow \infty$ (GS '06).

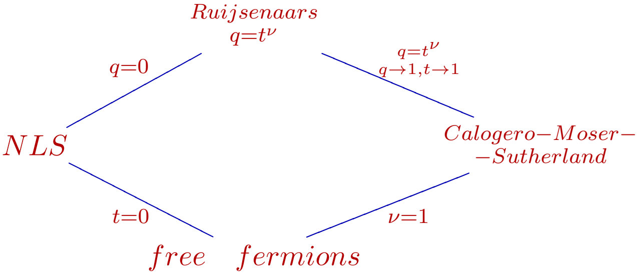
Partition function is sum over (Bethe equations):

$$e^{2\pi i\sigma_j(k+c_v)} \prod_{i\neq j} \frac{te^{2\pi i(\sigma_i-\sigma_j)} - 1}{te^{2\pi i(\sigma_j-\sigma_i)} - 1} = 1$$

These are Bethe equations for XXZ spin chain with spin s and in $s \rightarrow -i\infty$ limit. Latter corresponds to supersymmetric vaua of $3d$ $N = 2$ gauge theory (form the list shown earlier) on $R^2 \times S^1$.

For elliptic case - Ω-background instead of KK. Elliptic version of Ruijsenaars '87 appears in 5d SYM on $(S^1 \times R^2_{\epsilon}) \times R^2$, connects to everything. What about 6d theory on $(T^2 \times R^2_{\epsilon}) \times R^2$?


$Mac(q,t)$
$q=0$
$q=t^\nu$,
$q\to 1, t\to 1$
HL
$t=1/p$
$GL(Q_p)$
$Jack$
$t=0$
$\nu=1$
$Schur$


$Ruijsenaars$
$q=t^\nu$
$q=0$
$q=t^\nu$
$q\to 1, t\to 1$
NLS
$Calogero-Moser-$
$-Sutherland$
$t=0$
$\nu=1$
$free \quad fermions$

4d SYM and Algebraic Integrable Systems

SW prepotential $\mathcal{F}(a)$ interpreted in terms of classical AIS - pToda, eCM (GKMMM '95, MW '95, DW '95):

• A complex algebraic manifold M^{2N} of complex dimension $2N$ with non-degenerate, closed holomorphic (2, 0)-form $\Omega_C^{2,0}$

• A holomorphic map $H : M \to C^N$, fibers $J_h = H^{-1}(h)$ are (polarized) abelian varieties (complex tori), $\{H_i, H_j\} = 0$

Polarization - Kahler form ω whose restriction on each fiber is integral class: $[w] \in H^2(J_h, Z) \cap H^{1,1}(J_h)$. $< A_i, B^j >= \delta_i^j$, basis in $H_1(J_h, Z)$. "Action variables":

$$a_i = \int_{A_i} \Theta_C, \qquad a_D^i = \int_{B^i} \Theta_C, \qquad \Omega_C^{2,0} = d\Theta_C$$

Twice as many as the dimension of base - they must be related:

$$a_D^i = \frac{\partial \mathcal{F}(a)}{\partial a^i}; \quad \theta = \sum_{i=1}^{N} a_D^i da_i = d\mathcal{F}(a)$$

$\mathcal{N}=2$ gauge theory on 2d Ω-background $R^2 \times R^2_\epsilon$ is a deformation of $\mathcal{N}=2$ theory on $R^2 \times R^2$ with one, equivariant, parameter ϵ which corresponds to the rotation of second R^2 around its origin.

Only 2d super-Poincare invariance is unbroken, four Q's. The effective theory is 2d with four supercharges. Alternative to KK.

NS '09: As such it has 2d effective $\mathcal{W}^{eff}$; computed as a limit of the partition function $\mathcal{Z}(\{a\}, \epsilon_1, \epsilon_2)$ in general Ω-background $R^2_{\epsilon_1} \times R^2_{\epsilon_2}$, e.g. for $N=2^*$ theory (eCM; $q=e^{i\tau}; \tau = i/g^2 + \theta$):

$$\mathcal{W}^{eff}(a;q,m,\epsilon) = \lim_{\epsilon_2 \to 0} \epsilon_2 log \mathcal{Z}(a;q,m,\epsilon,\epsilon_2) = \frac{\mathcal{F}_{eCM}(a;q,m)}{\epsilon} + \ldots$$

$$\mathcal{W}^{eff}(a;q,m,\epsilon) = \mathcal{W}_{pert}(a;\tau,m,\epsilon) + \mathcal{W}_{inst}(a;q,m,\epsilon)$$

$$\exp\left(\frac{\partial \mathcal{W}_{\rm pert}}{\partial a_i}\right) = e^{\frac{\pi i \tau a_i}{\epsilon}} \prod_{j \neq i} S(a_i - a_j); \quad S(x) = \frac{\Gamma\left(\frac{-m+x}{\epsilon}\right)}{\Gamma\left(\frac{-m-x}{\epsilon}\right)} \frac{\Gamma\left(1 - \frac{x}{\epsilon}\right)}{\Gamma\left(1 + \frac{x}{\epsilon}\right)}$$

$$\mathcal{W}_{inst}(a) = \int \mathrm{d}x \left[-\frac{\chi(x)}{2} \log\left(1 - qQ(x)e^{-\chi(x)}\right) + \mathrm{Li}_2\left(qQ(x)e^{-\chi(x)}\right)\right]$$

$$\chi(x) = \int \mathrm{d}y\, G_0(x-y) \log\left(1 - qe^{-\chi(y)}Q(y)\right)$$

$$G_0(x) = \partial_x \log \frac{(x+\epsilon)(x+m)(x-m-\epsilon)}{(x-\epsilon)(x-m)(x+m+\epsilon)}$$

$$Q(x) = \frac{P(x-m)P(x+m+\epsilon)}{P(x)P(x+\epsilon)}; \quad P(x) = \prod_{l=1}^{N}(x-a_l)$$

Energy spectrum of properly quantized system:

$$E_2 = \epsilon q \frac{\partial}{\partial q}\mathcal{W}^{eff}(a;q,m,\epsilon) = \epsilon \frac{\partial}{\partial \tau}\mathcal{W}^{eff}(a;q,m,\epsilon)$$

Evaluated on solutions of:

$$\frac{1}{2\pi i}\frac{\partial \mathcal{W}^{eff}(a;q,m,\epsilon)}{\partial a^i} = n_i$$

What is the meaning of this $\mathcal{W}^{eff}(a; q, m, \epsilon)$ (YY-function) in terms of the geometry of classical AIS?

Answered in RNS '11, with the help of NW '10 interpretation of above quantization and work on many body systems from '80-'90's.

Important example - Hitchin integrable system on $\Sigma_{g,n}$:

$$F_{z\bar{z}}(A) = [\Phi_z, \Phi_{\bar{z}}]$$
$$\nabla_z(A)\Phi_{\bar{z}} = 0$$
$$\nabla_{\bar{z}}(A)\Phi_z = 0$$

modulo unitary gauge transformations :

$$A \to g^{-1}Ag + g^{-1}dg; \quad \Phi \to g^{-1}\Phi g$$

Moduli space of solutions to Hitchin equations - phase space of algebraic integrable system. It is hyperkahler.

$g = 1, n = 1$, $G = U(N)$: N-particle class. eCM $\Leftrightarrow N = 2^*$ SYM.

Complex structure I - holomorphic coordinates $(A_z, \Phi_{\bar{z}})$. Depends on the choice of complex structure on $\Sigma_{g,n}$:

$$\Omega_I^{2,0} = \int_{\Sigma_{g,n}} \delta A_z \wedge \delta \Phi_{\bar{z}}$$

Poisson commuting H_i's for PGL_2 (μ_i: $3g-3+n$ Beltrami diffs):

$$H_i = \int_{\Sigma_{g,n}} \mu_i tr \Phi_z^2$$

$\Sigma_{g=0,n}$ - Hitchin H_i's $=$ Gaudin Hamiltonians.

Pick complex structure J (replace G by ${}^L G$) - holomorphic coord. $(A_z + i\Phi_z, A_{\bar{z}} + i\Phi_{\bar{z}})$; independent of complex structure on $\Sigma_{g,n}$:

$$\Omega_J^{2,0} = \int_{\Sigma_{g,n}} \delta A^c \wedge \delta A^c; \qquad A^c = A + i\Phi$$

In complex structure J Hitchin moduli space is the moduli space of G^C flat connections modulo complexified gauge transformations:

$$\{ \quad F(A+i\Phi) = 0 \quad / \quad G^c \quad gauge \quad transformations\}$$

For ${}^{L}G = SL(2,C)$ - fix some reference complex structure on $\Sigma_{g,n}$, local coordinates $(w,\bar{w})$ and describe generic complex structure via Beltrami diffs $\mu = \mu_{\bar{w}}^{w} d\bar{w}\partial_w$; pick a gauge:

$$A_{\bar{w}} - \mu A_w = \begin{pmatrix} -\frac{1}{2}\partial\mu & 0 \\ -\frac{1}{2}\partial^2\mu & \frac{1}{2}\partial\mu \end{pmatrix}, \; A_w = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}$$

where T obeys the compatibility condition (from flatness):

$$\left(\bar{\partial} - \mu\partial - 2\partial\mu\right) T = -\frac{1}{2}\partial^3\mu$$

Now

$$\Omega_J^{2,0} = \int_{\Sigma_{g,n}} \delta\mu \wedge \delta T$$

SL_2 oper - 2nd order diff. operator, acting on -1/2 differentials:

$$\mathcal{D} = -\partial^2 + T(z)$$

G-opers can be defined for general surface with punctures, where opers develop poles - here we consider only regular singularities.

Restrict to $g = 0$ with n marked points.

$$T(z) = \sum_{a=1}^{n} \frac{\Delta_a}{(z - x_a)^2} + \sum_{a=1}^{n} \frac{\epsilon_a}{z - x_a}$$

Δ_a are fixed and ϵ_a obey ($\Omega_J^{2,0} = \sum_{a=1}^{n} \delta\epsilon_a \wedge \delta x_a$):

$$\sum_{a=1}^{n} \epsilon_a = 0$$

$$\sum_{a=1}^{n} (x_a \epsilon_a + \Delta_a) = 0$$

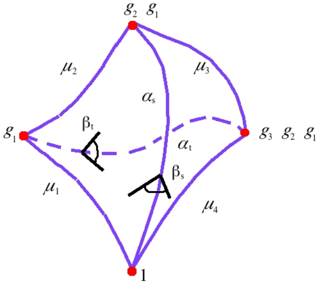
$$\sum_{a=1}^{n} \left(x_a^2 \epsilon_a + 2x_a \Delta_a\right) = 0$$

Fix complex structure (x_a's); space of opers, parametrized by ϵ_a, is a Lagrangian submanifold in the moduli space of flat connections.

One can introduce other, topological, Darboux coordinates.

RNS '11: *YY-function is essentially the generating function of this Lagrangian submanifold in the special Darboux coord* (α_a, β_a).

$$\beta_a = \frac{\partial \mathcal{W}^{eff}(\{\alpha_a\},\{x_a\})}{\partial \alpha_a}; \quad \epsilon_a = \frac{\partial \mathcal{W}^{eff}(\{\alpha_a\},\{x_a\})}{\partial x_a}$$



g_i - monodromies around each point, $tr g_i = m_i$ fixed, and (Δ_i, μ_i) expressed in m_i. Darboux variables (α_s, β_s) ((α_t, β_t)) correspond to "s-chanel" ("t") degenerations.

From the point of view of AGT relation this is a classical limit in CFT, so one should see it purely in CFT language (Teschner '10). For special values of m_i such formulas were seen before in Liouville theory (Zam.-Zam. '95, Takhtajan-Zograf '88); in the approach of RNS '11 it should correspond to the particular choice of real slice.