Integrability in Quantum Theory, and Applications

Samson L. Shatashvili

Trinity College, Dublin & IHES, Bures-Sur-Yvette
Supersymmetric vacua of gauge theories with four supercharges ⇔ Bethe eigenstates and excitation spectrum of integrable lattice models, Hitchin systems, its limits (quantum many body systems)

- Thermodynamic Bethe ansatz (TBA) type of equations, developed for quantum integrable systems, play the central role

- TBA type equations appear in the study of wall-crossing phenomena in counting of BPS states in $\mathcal{N}=2$ theories

- Correspondence between 4d instanton calculus and two 2d CFT has important consequences, both for CFT and gauge theory

- TBA type equations appear in computing the amplitudes and the expect. values of Wilson and ’t Hooft loops for maximal SUSY

- Quantum integrability is central in the study of maximally supersymmetric gauge theories in four dimensions when computing the anomalous dimensions, and in AdS/CFT correspondence
• The spectrum of the equivariant Donaldson theory and its generalizations ⇔ the spectrum of the quantized SW theory

• Partition functions of closed topological strings ⇔ the tau-functions of classical integrable hierarchies, and the inclusion of open strings connects to quantum integrability

• Dimer models and their applications to the topological strings on the toric Calabi-Yau manifolds links to the quantum integrability

• Geometric Langlands correspondence, its quantum field theory realization, and the possibility to reach out to number theory

• SLE, random growth and matrix models, emergent geometry

• Connections and inter-relations with representation theory

• The integrable QFT’s in 1+1 dimensions (sine-Gordon, etc.)

• Theory of solitons ⇔ Classical Inverse Scattering Method, Lax pairs, Spectral curves, etc. and quantization
NS '09: For every quantum integrable system, solved by BA, there is a SUSY gauge theory with 4 supercharges, $Q_+, Q_-, Q_+, Q_-$. s.t.:

a) exact Bethe eigenstates correspond to SUSY vacua,
b) ring of commuting Hamiltonians $\iff$ (twisted) chiral ring.

- The effective twisted superpotential $\iff$ Yang-Yang function

\[
\tilde{W}_{\text{eff}}(\sigma) = Y(\lambda)
\]

\[
\sigma_i = \lambda_i; \quad i = 1, \ldots, N; \quad G = U(N)
\]

- VEV of chiral ring operators $O_k$ $\iff$ Energies:

\[
<\lambda|O_k|\lambda> = E_k(\lambda)
\]

\[
H_k \Psi(\lambda) = E_k(\lambda) \Psi(\lambda)
\]

Vacua/Bethe Equations - critical points of $\tilde{W}_{\text{eff}}(\sigma)/Y(\lambda)$ as functions of abelian components of scalar field $\sigma_i$ / rapidities $\lambda_i$. 
What are these quantum integrable systems?

After massive fields (2d) are integrated out chiral ring generators are invariant functions of \( \Sigma = \sigma + \ldots \) on Coulomb branch.

SUSY vacua - there are two options: 1. topological or 2. physical.

1. Topologically twisted (on cylinder) abelianized theory has the action completely determined by \( \tilde{W}^{\text{eff}}(\sigma) \) of physical theory:

\[
S_{\text{top}} = \int \left[ \frac{\partial \tilde{W}^{\text{eff}}(\sigma)}{\partial \sigma_i} F^i(A) + \frac{\partial^2 \tilde{W}^{\text{eff}}(\sigma)}{\partial \sigma_i \partial \sigma_j} \lambda^i \wedge \lambda^j \right]
\]

compare \( S_{2d-YM} = \int \left[ \sigma_i F^i(A) + \lambda^i \wedge \lambda^j \right] \)

Canonical quantization - momentum conjugate to the monodromy of abelian gauge field \( x^i = \int_{S^1} A^i \) is quantized:

\[
\frac{1}{2\pi i} \frac{\partial \tilde{W}^{\text{eff}}(\sigma)}{\partial \sigma^i} = n_i
\]
2. Physical: suppose we have the theory with the effective twisted superpotential $\tilde{W}^{\text{eff}}(\sigma)$ (abelianized).

The target space of the effective sigma model is disconnected, with $\vec{n}$ labeling the connected components (gauge flux quantization) with potential:

$$U_{\vec{n}}(\sigma) = \frac{1}{2} g^{ij} \left( -2\pi i n_i + \frac{\partial \tilde{W}^{\text{eff}}}{\partial \sigma^i} \right) \left( +2\pi i n_j + \frac{\partial \tilde{W}^{\text{eff}}}{\partial \bar{\sigma}^j} \right)$$

Now we need to find the minimum of potential - again:

$$\frac{1}{2\pi i} \frac{\partial \tilde{W}^{\text{eff}}(\sigma)}{\partial \sigma^i} = n_i$$

Or equivalently - SUSY vacua correspond to solution of equation:

$$\exp \left( \frac{\partial \tilde{W}^{\text{eff}}(\sigma)}{\partial \sigma^i} \right) = 1$$
• XXX spin chain - 2d gauge theory

For $SU(2)$, $s = \frac{1}{2}$ spin chain of length $L$ in $N$-particle sector $\Leftrightarrow U(N)$ 2d $N = 2$ gauge theory with $L$ fundamentals, $L$ anti-funds and 1 adjoint, with twisted masses $m_i$ and complexified $\theta$ term; $m_i$ are impurities $\mu_i$, $\theta$ - quasi-periodic boundary conditions, ...

• XXXZ spin chain - 3d gauge theory on $\mathbb{R}^2 \times S^1$

• XYZ spin chain - 4d gauge theory on $\mathbb{R}^2 \times T^2$

• Arbitrary spin group, representation, impurities, limiting models

• NLS, $N$-particles on $S^1$, $\delta$-function potential - 2d $\mathcal{N} = 4$ +...

• Periodic Toda - 4d pure $\mathcal{N} = 2$ theory on $\mathbb{R}^2 \times \mathbb{R}_\epsilon^2$

• Elliptic Calogero-Moser - 4d $\mathcal{N} = 2^*$ theory on $\mathbb{R}^2 \times \mathbb{R}_\epsilon^2$
Consider 2d pure $N = 4$ gauge theory ($G = U(N)$) broken down to $N = 2$ by the twisted mass ($m$) term for the adjoint chiral multiplet - $N = 2^*$. Add tree level twisted superpotential:

$$\tilde{W}(\sigma) = \frac{1}{2} tr\sigma^2$$

Vacuum equations:

$$e^{i\sigma_j} = \prod_{i=1}^{N} \frac{\sigma_i - \sigma_j + m}{\sigma_i - \sigma_j + m}$$

For $m = ic$, $c \in \mathbb{R}$, this is Bethe equation for NLS quantum theory in $N$-partical sector.

This is the first example treated in the topological field theory language in MNS ’97 and later in GS ’06–’07.
This topological theory, $YMH$ theory, computes equivariant intersection numbers on the moduli space of Higgs bundles introduced by Hitchin:

$$F_{z\bar{z}}(A) = [\Phi_z, \Phi_{\bar{z}}]$$

$$\nabla_z(A)\Phi_{\bar{z}} = 0$$

$$\nabla_{\bar{z}}(A)\Phi_z = 0$$

modulo unitary gauge transformations:

$$A \rightarrow g^{-1}Ag + g^{-1}dg; \quad \Phi \rightarrow g^{-1}\Phi g$$

$z$ - local coordinates on Riemann surface, $\Phi$ - adjoint 1-form.

Moduli space of solutions to Hitchin equations - phase space of algebraic integrable system. It is hyperkahler. See later.

$NLS$ in $N$-particle sector is described by integrable system of $N$ non-relativistic particles on $S^1$ with $\delta$-function interactions.
\[ H_2 = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j) \]

Eigenvectors - spherical vectors in the representation theory of degenerate double affine Hecke algebra.

Latter is connected to the representation theory of \( GL(N, Q_p) \) - the wave functions are a limit of Hall-Littlewood polynomials, generalized zonal spherical functions for \( GL(N, Q_p) \):

\[
\prod_i \frac{1 - t}{1 - t^{m_i}} \sum_{w \in S_N} (-1)^{l(w)} w(\Lambda_1^{\mu_1} ... \Lambda_N^{\mu_N}) \prod_{i < j} \frac{\Lambda_i - \Lambda_j t}{\Lambda_i - \Lambda_j}
\]

\((\mu_1, ..., \mu_N)\) is a partition of length at most \( N \): \((1^{m_1}, .., r^{m_r}, ...)\).

\(NLS\) wave-functions correspond to analytic continuation with

\[
\mu_i = \frac{x_i}{\epsilon}, \quad \Lambda_i = e^{2\pi \epsilon \sigma_i}, \quad t = e^{2\pi i c \epsilon}, \quad \epsilon \to 0 \quad [p \to 1]
\]

This is continuous limit of discretized version of \( H_2 \).
GL(N, Q_p) zonal spherical functions are Macdonald’s \( M(q, t) \) for \( q = 0, t = p^{-1} \). Eigenfunctions of \( H_2 \) discretized (van Diejen, ’06).

\[ M(q, t = q^\nu) : \] relativistic Calogero-Moser-Sutherland (Ruijsenaars ’87) \( \rightarrow G/G \ WZW \), with Wilson lines (Gorsky, Nekrasov ’94).

There is another 2d (generalized) \( G/G \ WZW \) interpretation which has limit to \( YMH \) topological theory for \( k \rightarrow \infty \) (GS ’06).

Partition function is sum over (Bethe equations):

\[
e^{2\pi i \sigma_j (k+c_v)} \prod_{i \neq j} \frac{te^{2\pi i (\sigma_i - \sigma_j)} - 1}{te^{2\pi i (\sigma_j - \sigma_i)} - 1} = 1
\]

These are Bethe equations for \( XXZ \) spin chain with spin \( s \) and in \( s \rightarrow -i\infty \) limit. Latter corresponds to supersymmetric vaua of 3d \( N = 2 \) gauge theory (form the list shown earlier) on \( R^2 \times S^1 \).

For elliptic case - \( \Omega \)-background instead of \( KK \). Elliptic version of Ruijsenaars ’87 appears in 5d SYM on \( (S^1 \times R^2_\epsilon) \times R^2 \), connects to everything. What about 6d theory on \( (T^2 \times R^2_\epsilon) \times R^2 \)?
Mac($q, t$)

- $q=0$
- $q=t^\nu$, $q\rightarrow 1, t\rightarrow 1$

HL

- $t=1/p$
- $GL(Q_p)$

Jack

- $t=0$
- $\nu=1$

Schur

Ruijsenaars

- $q=t^\nu$

- $q=0$
- $q=t^\nu$, $q\rightarrow 1, t\rightarrow 1$

NLS

- $t=0$
- $\nu=1$

Calogero–Moser–Sutherland

free fermions
4d SYM and Algebraic Integrable Systems

SW prepotential $\mathcal{F}(a)$ interpreted in terms of classical AIS -
pToda, eCM (GKMM '95, MW '95, DW '95):

- A complex algebraic manifold $M^{2N}$ of complex dimension $2N$
  with non-degenerate, closed holomorphic $(2, 0)$-form $\Omega_C^{2,0}$

- A holomorphic map $H : M \to C^N$, fibers $J_h = H^{-1}(h)$ are
  (polarized) abelian varieties (complex tori), $\{H_i, H_j\} = 0$

Polarization - Kahler form $\omega$ whose restriction on each fiber is
integral class: $[\omega] \in H^2(J_h, Z) \cap H^{1,1}(J_h)$. $\langle A_i, B^j \rangle = \delta^j_i$,
basis in $H_1(J_h, Z)$. “Action variables”:

$$a_i = \int_{A_i} \Theta_C, \quad a_D^i = \int_{B^i} \Theta_C, \quad \Omega_C^{2,0} = d\Theta_C$$

Twice as many as the dimension of base - they must be related:

$$a_D^i = \frac{\partial \mathcal{F}(a)}{\partial a^i}; \quad \theta = \sum_{i=1}^{N} a_D^i da_i = d\mathcal{F}(a)$$
$\mathcal{N} = 2$ gauge theory on 2d $\Omega$-background $R^2 \times R^2_\epsilon$ is a deformation of $\mathcal{N} = 2$ theory on $R^2 \times R^2$ with one, equivariant, parameter $\epsilon$ which corresponds to the rotation of second $R^2$ around its origin.

Only 2d super-Poincare invariance is unbroken, four $Q$’s. The effective theory is 2d with four supercharges. Alternative to KK.

**NS ’09:** As such it has 2d effective $\mathcal{W}^{eff}$; computed as a limit of the partition function $\mathcal{Z}(\{a\}, \epsilon_1, \epsilon_2)$ in general $\Omega$-background $R^2_{\epsilon_1} \times R^2_{\epsilon_2}$, e.g. for $N = 2^*$ theory (eCM; $q = e^{i\tau}$; $\tau = i/g^2 + \theta$):

$$\mathcal{W}^{eff}(a; q, m, \epsilon) = \lim_{\epsilon_2 \to 0} \epsilon_2 \log \mathcal{Z}(a; q, m, \epsilon, \epsilon_2) = \frac{\mathcal{F}_{eCM}(a; q, m)}{\epsilon} + ...$$

$$\mathcal{W}^{eff}(a; q, m, \epsilon) = \mathcal{W}_{pert}(a; \tau, m, \epsilon) + \mathcal{W}_{inst}(a; q, m, \epsilon)$$

$$\exp\left(\frac{\partial \mathcal{W}_{pert}}{\partial a_i}\right) = e^{\frac{\pi i \tau a_i}{\epsilon}} \prod_{j \neq i} S(a_i - a_j); \quad S(x) = \frac{\Gamma\left(\frac{-m+x}{\epsilon}\right) \Gamma\left(1 - \frac{x}{\epsilon}\right)}{\Gamma\left(\frac{-m-x}{\epsilon}\right) \Gamma\left(1 + \frac{x}{\epsilon}\right)}$$
\[ W_{\text{inst}}(a) = \int dx \left[ -\frac{\chi(x)}{2} \log \left( 1 - qQ(x)e^{-\chi(x)} \right) + \text{Li}_2 \left( qQ(x)e^{-\chi(x)} \right) \right] \]

\[ \chi(x) = \int dy G_0(x - y) \log \left( 1 - qe^{-\chi(y)}Q(y) \right) \]

\[ G_0(x) = \partial_x \log \frac{(x + \epsilon)(x + m)(x - m - \epsilon)}{(x - \epsilon)(x - m)(x + m + \epsilon)} \]

\[ Q(x) = \frac{P(x - m)P(x + m + \epsilon)}{P(x)P(x + \epsilon)}; \quad P(x) = \prod_{l=1}^{N} (x - a_l) \]

Energy spectrum of properly quantized system:

\[ E_2 = \epsilon q \frac{\partial}{\partial q} \mathcal{W}^{\text{eff}}(a; q, m, \epsilon) = \epsilon \frac{\partial}{\partial \tau} \mathcal{W}^{\text{eff}}(a; q, m, \epsilon) \]

Evaluated on solutions of:

\[ \frac{1}{2\pi i} \frac{\partial \mathcal{W}^{\text{eff}}(a; q, m, \epsilon)}{\partial a^i} = n_i \]
What is the meaning of this $\mathcal{W}^{eff}(a; q, m, \epsilon)$ ($YY$-function) in terms of the geometry of classical AIS?

Answered in RNS ’11, with the help of NW ’10 interpretation of above quantization and work on many body systems from ’80-’90’s.

Important example - Hitchin integrable system on $\Sigma_{g,n}$:

$$F_{z\bar{z}}(A) = [\Phi_{z}, \Phi_{\bar{z}}]$$

$$\nabla_{z}(A)\Phi_{\bar{z}} = 0$$

$$\nabla_{\bar{z}}(A)\Phi_{z} = 0$$

modulo unitary gauge transformations :

$$A \rightarrow g^{-1}Ag + g^{-1}dg; \quad \Phi \rightarrow g^{-1}\Phi g$$

Moduli space of solutions to Hitchin equations - phase space of algebraic integrable system. It is hyperkähler.

$g = 1, n = 1, G = U(N)$: $N$-particle class. eCM $\Leftrightarrow N = 2^* \text{ SYM.}$
Complex structure $I$ - holomorphic coordinates $(A_z, \Phi \overline{z})$. Depends on the choice of complex structure on $\Sigma_{g,n}$:

$$\Omega_{I}^{2,0} = \int_{\Sigma_{g,n}} \delta A_z \wedge \delta \Phi \overline{z}$$

Poisson commuting $H_i$'s for $PGL_2$ ($\mu_i$: $3g-3+n$ Beltrami diffs):

$$H_i = \int_{\Sigma_{g,n}} \mu_i \text{tr} \Phi^2 \overline{z}$$

$\Sigma_{g=0,n}$ - Hitchin $H_i$'s = Gaudin Hamiltonians.

Pick complex structure $J$ (replace $G$ by $L^G$) - holomorphic coord. $(A_z + i\Phi_z, A_{\overline{z}} + i\Phi_{\overline{z}})$; independent of complex structure on $\Sigma_{g,n}$:

$$\Omega_{J}^{2,0} = \int_{\Sigma_{g,n}} \delta A^c \wedge \delta A^c; \quad A^c = A + i\Phi$$

In complex structure $J$ Hitchin moduli space is the moduli space of $G^C$ flat connections modulo complexified gauge transformations:

$$\{ \quad F(A + i\Phi) = 0 \quad / \quad G^C \quad gauge \quad transformations \}$$
For $L G = SL(2, C)$ - fix some reference complex structure on $\Sigma_{g,n}$, local coordinates $(w, \bar{w})$ and describe generic complex structure via Beltrami diffs $\mu = \mu^w_w d\bar{w} \partial_w$; pick a gauge:

$$A\bar{w} - \mu A_w = \begin{pmatrix} -\frac{1}{2} \partial \mu & 0 \\ -\frac{1}{2} \partial^2 \mu & \frac{1}{2} \partial \mu \end{pmatrix}, \quad A_w = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}$$

where $T$ obeys the compatibility condition (from flatness):

$$(\bar{\partial} - \mu \partial - 2\partial \mu) T = -\frac{1}{2} \partial^3 \mu$$

Now

$$\Omega^2;0 = \int_{\Sigma_{g,n}} \delta \mu \wedge \delta T$$

$SL_2$ oper - 2nd order diff. operator, acting on -1/2 differentials:

$$\mathcal{D} = -\partial^2 + T(z)$$

$G$-opers can be defined for general surface with punctures, where opers develop poles - here we consider only regular singularities.
Restrict to \( g = 0 \) with \( n \) marked points.

\[
T(z) = \sum_{a=1}^{n} \frac{\Delta_a}{(z - x_a)^2} + \sum_{a=1}^{n} \frac{\epsilon_a}{z - x_a}
\]

\( \Delta_a \) are fixed and \( \epsilon_a \) obey \( (\Omega_{j}^{2,0} = \sum_{a=1}^{n} \delta \epsilon_a \wedge \delta x_a) \):

\[
\sum_{a=1}^{n} \epsilon_a = 0
\]

\[
\sum_{a=1}^{n} (x_a \epsilon_a + \Delta_a) = 0
\]

\[
\sum_{a=1}^{n} (x_a^2 \epsilon_a + 2x_a \Delta_a) = 0
\]

Fix complex structure (\( x_a \)'s); space of opers, parametrized by \( \epsilon_a \), is a Lagrangian submanifold in the moduli space of flat connections. One can introduce other, topological, Darboux coordinates.

**RNS '11:** *YY-function is essentially the generating function of this Lagrangian submanifold in the special Darboux coord \((\alpha_a, \beta_a)\).*
\[ \beta_a = \frac{\partial W_{\text{eff}}(\{\alpha_a\}, \{x_a\})}{\partial \alpha_a}; \quad \epsilon_a = \frac{\partial W_{\text{eff}}(\{\alpha_a\}, \{x_a\})}{\partial x_a} \]

\( g_i \) - monodromies around each point, \( \text{tr} g_i = m_i \) fixed, and \((\Delta_i, \mu_i)\) expressed in \( m_i \). Darboux variables \((\alpha_s, \beta_s)\) \((\alpha_t, \beta_t)\) correspond to “s-chanel” (“t”) degenerations.

From the point of view of AGT relation this is a classical limit in CFT, so one should see it purely in CFT language (Teschner ’10). For special values of \( m_i \) such formulas were seen before in Liouville theory (Zam.-Zam. ’95, Takhtajan-Zograf ’88); in the approach of RNS ’11 it should correspond to the particular choice of real slice.