Knot Invariants From Maximally Supersymmetric Yang-Mills Theory

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I decided that rather than any technical details, I would give an overview of the content of several recent papers. I won't try to give references to all the basic results that I will mention along the way, but I should at least mention the paper by S. Gukov, A. Schwarz and C. Vafa, hep-th/0412243, which was part of the inspiration.

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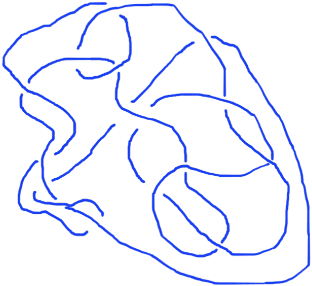

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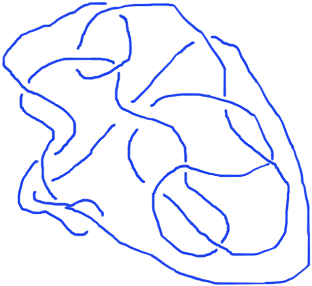

There is nothing like that in four dimensions.

To get something interesting, we are going to consider $\mathcal{N} = 4$ super Yang-Mills theory not on $\mathbb{R}^4$ but on a half-space $\mathbb{R}^3 \times \mathbb{R}_+$, where $\mathbb{R}_+$ is a half-line $y \geq 0$.

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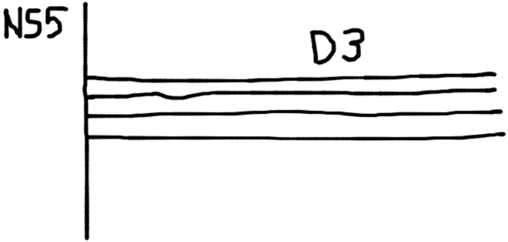

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So we can do topological field theory on $\mathbb{R}^3 \times \mathbb{R}_+$ in this situation with Wilson operators for an arbitrary K.Their expectation values are topological invariants, but not interesting, for the same reason as before.

We actually do get something interesting if we take the gauge theory θ-angle to be nonzero. The D3-NS5 boundary condition (which was generalized to this situation in D. Gaiotto and EW, arXiv:0804.2902) still preserves 8 supersymmetries, but a different 8.

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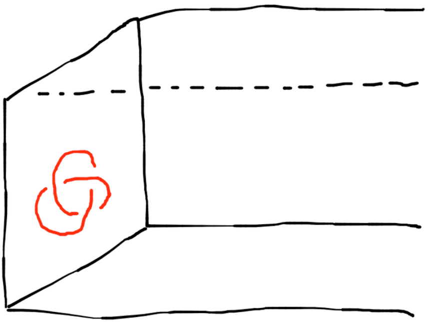

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If one applies supersymmetric localization in this situation, one learns something interesting: the expectation value of one of these Wilson operators in the boundary of a four-dimensional space can be computed in a purely three-dimensional topological field theory, namely (bosonic) Chern-Simons theory.

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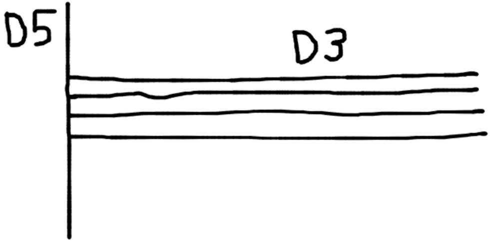

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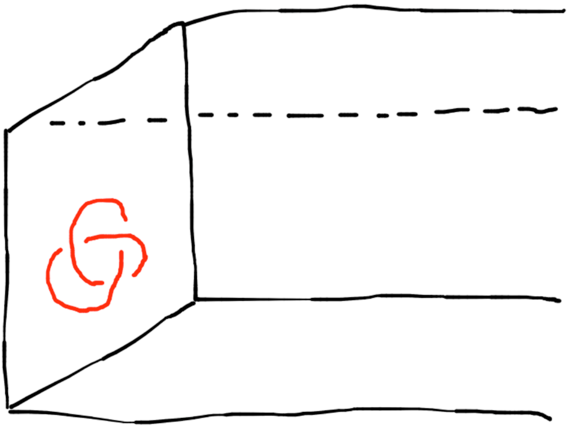

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(These equations were introduced by Kapustin and me in studying geometric Langlands. They have also been used in K. Lee and H. Yee, hep-th/0606159 to discuss six-dimensional string webs.) Localization on the solutions of an equation is the simplest sort of answer that one sometimes gets from supersymmetric localization.

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$$\frac{1}{8\pi^2}\int_{\mathbb{R}^3\times\mathbb{R}_+} \mathrm{Tr}\, F\wedge F$$

is equal to n. Then the path integral Z is

$$Z=\sum_n a_n q^n,$$

where in the purely 3d description by Chern-Simons theory,

$$q=\exp(2\pi i/(k+2)).$$

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In the D4-brane description, the knot is still represented by an 't Hooft operator (which now is supported on $K \times S^1$, where S^1 is the circle that was generated by the T-duality).

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$$Z = \mathrm{Tr}\,(-1)^F q^P,$$

where P is the instanton number.

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(Also, the states carry an integer-valued fermion number F, and the index only depends on $(-1)^F$, i.e. on the values of F mod 2. And introducing a fifth dimension lets us consider more general 't Hooft operators.)

So there is a more powerful theory: we just study the space $\mathcal{H}$ of physical states, instead of the index.

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The D4-brane gauge theory isn't ultraviolet complete, but it has a well-known ultraviolet completion in the M5-brane system, or more exactly in the six-dimensional (0,2) superconformal field theory. The whole construction can be usefully expressed in six-dimensional terms. The basic idea here is that one just replaces the half-line $\mathbb{R}_+$ of the D4-brane worldvolume by a copy of $\mathbb{R}^2$ with a "cigar"-like metric:

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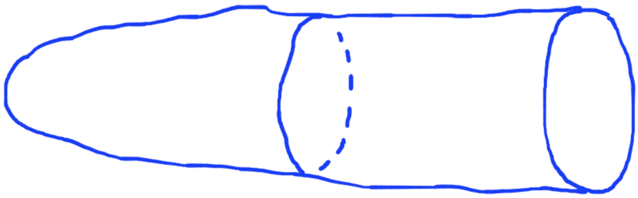

The cigar, which I will call D, is a cylinder of revolution. If one reduces the M5-brane theory on the $U(1)$ orbits, the M5-brane theory is replaced by a D4-brane theory, and D is replaced by

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This leads to the $\mathbb{R}_+$ factor in the D3-NS5, D3-D5, and D4-D6 descriptions.

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