$\mathcal{N} = 2$ gauge theories and algebras

- **S.C., Andrew Neitzke & Cumrun Vafa,** `arXiv: 1006.3435`.
- **S.C., & Cumrun Vafa,** `arXiv: 1103.5832`.
- **Murad Alim, S.C., Clay Cordova, Sam Espahbodi, Aswhin Rastogi,& Cumrun Vafa,** `to appear & in progress`.

An important class of $\mathcal{N} = 2$ $4d$ models:

Complete (quiver) theories

QUIVER $\mathcal{N} = 2$ THEORIES

★ There is a large class of $4d$ $\mathcal{N} = 2$ theories whose BPS states correspond to the supersymmetric states of a $1d$ *quiver* SQM

$$D^a = F_\alpha = 0. \qquad (*)$$

• Douglas, Moore • Douglas *et al* • Denef

★ from a $4d$ viewpoint the quiver Q encodes the electric/magnetic/flavor charges and their Dirac pairing

- $\Gamma \simeq \oplus_{i=1}^{r} \mathbb{Z}\, e_i$ **lattice of (quantized) electric/magnetic/flavor charges**
- $\langle \gamma, \gamma' \rangle_{\mathrm{Dirac}} = -\langle \gamma', \gamma \rangle_{\mathrm{Dirac}} \in \mathbb{Z}, \quad \gamma, \gamma' \in \Gamma$ (*bilinear*)
- **associate a quiver** Q : one node i per lattice generator e_i.

  the integer $\langle e_i, e_j \rangle_{\mathrm{Dirac}}$ gives the *signed* number of arrows $i \to j$

★ $1d$ gauge group $\prod_i U(N_i) \Rightarrow$ BPS charge vector $\sum_i N_i\, e_i \in \Gamma$

★ **central charge** $Z(\cdot) \colon \Gamma \to \mathbb{C}$ (linear). $M = |Z(\gamma)|$, $\gamma \in \Gamma$.

★ BPS states correspond to Z–stable representations X of the quiver path algebra subjected to the relations $F_\alpha \equiv \partial_\alpha W = 0$. In particular, $\mathrm{End}\, X = \mathbb{C}$ (bricks).

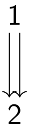
★ The quiver Q is **not** unique: $1d$ Seiberg duality.

Quiver mutations: products of basic mutations μ_k at k–th node

μ_k :
1) invert all arrows trough node k;
2) for each pairs of arrows $i \xrightarrow{\alpha} k \xrightarrow{\beta} j$, add a new arrow $i \xrightarrow{[\alpha\beta]} j$;
3) delete pair of opposite arrows $i \leftrightarrows j$;
4) replace the superpotential $W \to \mu_k(W)$.

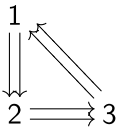
Two quivers are *mutation equivalent* if related by a chain of Seiberg dualities. A (quiver) $\mathcal{N} = 2$ theory is associated to a **mutation class** of 2–acyclic quivers.

EXAMPLES:



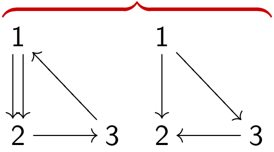
'Kronecker quiver'

$SU(2)$ SYM



'Markov quiver'

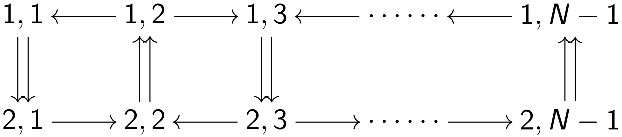
$SU(2)$ $\mathcal{N} = 2^*$



$SU(2)$ SQCD $N_f = 1$

(2 distinct quivers in class)

- Kronecker quiver $\equiv$ the acyclic affine quiver $\widehat{A}_1$
- $SU(2)$ SQCD $N_f = 1 \equiv$ the acyclic affine quiver $\widehat{A}_2$



$SU(N)$ SYM *(just a representative in a big class)*

The central charge $Z_i \equiv Z(e_i)$ is a function of all physical parameters (gauge couplings, masses, Coulomb branch coordinates,...)

Changing the parameters, we cross walls of marginally stability: the BPS spectrum jumps $\Rightarrow$ the Kontsevich–Soilbelman WCF

We also need to change the quiver to another one in the mutation–class

NOT all the consequences of the WCF are physical
NOT all the (mathematical) formal BPS chambers $\mathscr{C}_i^{\mathrm{BPS}}$ exist:

The image of the map ($r = \mathrm{rank}\,\Gamma$)

$$\mathscr{P} \equiv (\text{space of physical parameters}) \longrightarrow$$
$$\longrightarrow (\text{space of central charges}) \equiv \mathbb{C}^r = \bigcup_i \overline{\mathscr{C}_i^{\mathrm{BPS}}},$$

has, in general, a *positive codimension* and some chambers $\mathscr{C}_i^{\mathrm{BPS}}$ are **not physically realizable**

$$\mathrm{im}\,\mathscr{P} \cap \mathscr{C}_i^{\mathrm{BPS}} = \emptyset$$

One has to determine $\mathrm{im}\,\mathscr{P} \subset \mathbb{C}^r$. Simpler case: *complete theories*

Complete (quiver) $\mathcal{N} = 2$ theories: the map $\mathscr{P} \to \mathbb{C}^r$ is 'generically surjective'

To get a **rough** idea of what *completeness* is about, consider a theory with a weakly–coupled Lagrangian description **(most complete theories have NO such description !)**

Count of dimensions

- $\mathrm{rank}\,\Gamma = \#\text{electric} + \#\text{magnetic} + \#\text{flavor} = 2\,\mathrm{rank}\,G + \mathrm{rank}\,G_f$
- $\dim(\text{parameter space}) =$

$= \#\text{gauge couplings} + \#\dim(\text{Coulomb branch}) + \#\text{masses} =$

$= \#(\text{simple factors of } G) + \mathrm{rank}\,G + \mathrm{rank}\,G_f$

We get the condition $\mathrm{rank}\,G = \#(\text{simple factors of } G) \Rightarrow$

$$G = SU(2)^m, \qquad m \in \mathbb{N}$$

Why complete theories are interesting?

- they can be completely classified;
- many interesting models turn out to be complete;
- to compute their non–perturbative physics (not just the BPS spectrum !) in **detail** is both easy and elegant;
- it is an ideal 'non–perturbative laboratory': the insights we get allow us to extend methods and results to more general $\mathcal{N} = 2$ theories (general G,...)

Remark: complete theories are, in particular, UV complete.

CLASSIFICATION: The idea

If the theory is *complete*, we have enough physical deformations to give a large mass M to all states, but those having a charge vector of the form

$$n\, e_i + m\, e_j$$

for any chosen pair of nodes i, j of the quiver Q. In the decoupling limit $M \to \infty$ we get an effective $\mathcal{N} = 2$ theory with $\mathrm{rank}\, \Gamma_{\text{eff}} = 2$ and an **effective** quiver

$$Q_{\text{eff}}: \qquad i \xrightarrow{\langle e_i, e_j \rangle} j$$

- 2–nodes quivers are consistent with QFT iff have at most two arrows.
- Complete $\Rightarrow$ all quivers in its class correspond to physical regimes

The quiver of a *complete* theory has the property that all quivers in its class have *at most 2 arrows* between *any pair* of nodes

Quivers with this property are classified: Felikson–Shapiro–Tumarkin thm

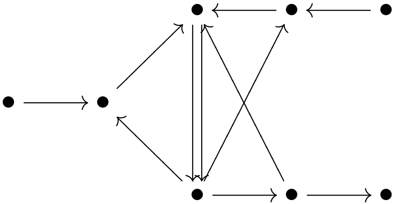
1. all incidence quivers of ideal triangulations of bordered surfaces with punctures and marked points on the boundary
2. 11 **exceptional** mutation classes:
   - E_6, E_7, E_8 — E–type *finite–type* Dynkin graph
   - $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ — E–type *affine* Dynkin graph
   - $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ — E–type *elliptic* (toroidal) Dynkin graph
   - X_6, X_7 — Derksen–Owen quivers

   with all triangles oriented

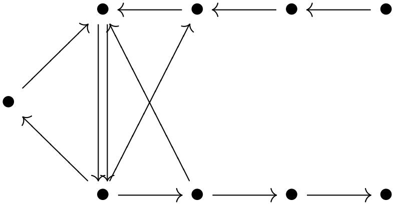
All these quiver classes correspond to $\mathcal{N} = 2$ theories which may be string–engineered or have a Lagrangian description

Full list of the complete quiver $\mathcal{N} = 2$ theories

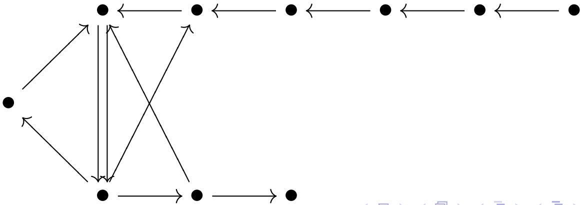
$E_6^{(1,1)}$:



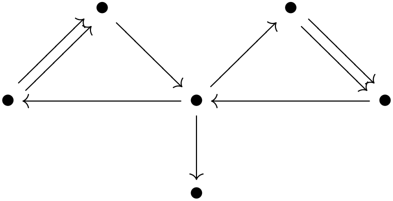
$E_7^{(1,1)}$:



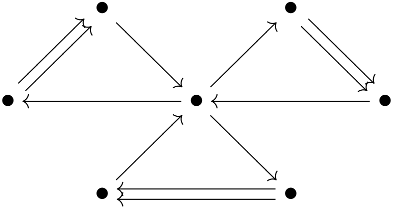
$E_8^{(1,1)}$:



X_6 :



X_7 :



Incidence quiver of an ideal triangulation

- nodes of quiver $\leftrightarrow$ arcs γ_ℓ of the ideal triangulation
- $\#$ arrows from i to j

$$\#\{i \to j\} = \sum_{\substack{\triangle \text{ shared} \\ \text{by arcs } \gamma_i, \gamma_j}} \pm 1 \quad \Rightarrow \quad |\#\{i \to j\}| \leq 2$$

Ideal triangulations of a surface have *mutation equivalent* quivers

$\Rightarrow$ **complete**

- Number of arcs $\equiv \mathrm{rank}\,\Gamma$

$$r = 6g - 6 + 3p + 3b + c$$

g genus
p # punctures
b # boundary components
c # boundary marks

each boundary component has at least one mark, and $p + b > 0$

IDENTIFICATION WITH THE $\mathcal{N}=2$ MODELS

Quiver of an ideal triangulation of a surface with given $(g, p, b, c_i) \Leftrightarrow$ $\mathcal{N}=2$ model constructed *á la* Gaiotto–Moore–Neitzke: A_1 $(2,0)$ theory on the a curve of genus g, the Hitchin quadratic differential ϕ_2 has an ordinary double pole for each puncture, a pole of order $k+2$ for a boundary component with k marked points

UV superconformal $\leftrightarrow \begin{cases} g=0 \ p=0,1, \ \ b=1 & \text{\textit{A,D}–type AD} \\ b=0 & \text{Gaiotto theories} \end{cases}$

X_7 is a mass deformation of the $g=2$, $p=b=0$ model (hence UV superconformal). X_6 is a decoupling limit of X_7, (hence AF)

All other complete theories have quivers which are (mutation equivalent) to Dynkin diagrams

The $\mathcal{N} = 2$ $4d$ models associated to Lie algebras

(In particular, the *nine* E-type exceptional complete theories)

Complete theories with Dynkin quiver

- simply–laced Dynkin quivers $\Rightarrow$ *ADE* Argyres–Douglas models **vector–less, UV superconformal**;
- *affine* Dynkin quivers $\Rightarrow \widehat{A}(p,q)$ $(p \geq q \geq 1)$, $\widehat{D}_r$ $(r \geq 4)$, and $\widehat{E}_r$ $(r = 6,7,8)$ **asymptotically–free** $SU(2)$ **gauge theories**;
- elliptic (toroidal) Dynkin quivers $\Rightarrow D_4^{(1,1)} \equiv SU(2)$ SQCD $N_f = 4$, $E_r^{(1,1)}$ $(r = 6,7,8)$ **UV superconformal** $SU(2)$ **gauge theories**

The BPS spectrum has a Lie algebraic interpretation

- charge lattice $\equiv$ root lattice of corresponding Lie algebra
- $\alpha \in \Gamma$ is the the charge vector of a stable BPS particle $\Rightarrow \alpha$ is a (brick) root of the (finite–dimensional, affine, or toroidal) Lie algebra
    - *real* root $\Rightarrow$ hypermultiplet
    - *imaginary* root $\Rightarrow$ vector–multiplet
- Kac–Moody (finite or affine): $\exists$ strong coupling chamber with just BPS hypermultiplets of charge vectors α_i (simple roots)

But for the Argyres–Douglas models, all these theories have *weakly coupled chambers* with a W BPS vector–multiplet. In the $g \to 0$ limit we get a theory of an $SU(2)$ SYM weakly gauging the global $SU(2)$ symmetries of a collection of D–type Argyres–Douglas systems:

Dynkin quiver class	AD 'matter' coupled to $SU(2)$ SYM
$\widehat{A}(p,q),\ p \geq q \geq 1$	$D_p,\ D_q$ (E.g. $SU(2)$ SQCD $N_f = 0,1,2$)
$\widehat{D}_r,\ r \geq 4$	$D_2,\ D_2,\ D_{r-2}$ (E.g. $SU(2)$ SQCD $N_f = 3$)
$\widehat{E}_r,\ r = 6,7,8$	$D_2,\ D_3,\ D_{r-3}$
$D_4^{(1,1)}$	$D_2,\ D_2,\ D_2,\ D_2$ $SU(2)$ SQCD $N_f = 4$
$E_6^{(1,1)}$	$D_3,\ D_3,\ D_3$
$E_7^{(1,1)}$	$D_2,\ D_4,\ D_4$
$E_8^{(1,1)}$	$D_2,\ D_3,\ D_6$

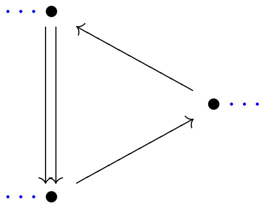
Convention: D_1 is the empty matter, D_2 is a fundamental hypermultiplet

$$b = 4 - 2\sum_i \left(1 - \frac{1}{r_i}\right)$$

Simpler technique: graphical analysis *S.C., C. Vafa*, 1103.5832

A quiver with a Kronecker subquiver $i \rightrightarrows j$ has a stable BPS vector–multiplet of charge $e_i + e_j$ iff $\mathrm{Im}\, Z(e_j) < \mathrm{Im}\, Z(e_i)$

The gauge coupling of a system with fundamental (electric) charge to a vector–multiplet from a Kronecker subquiver



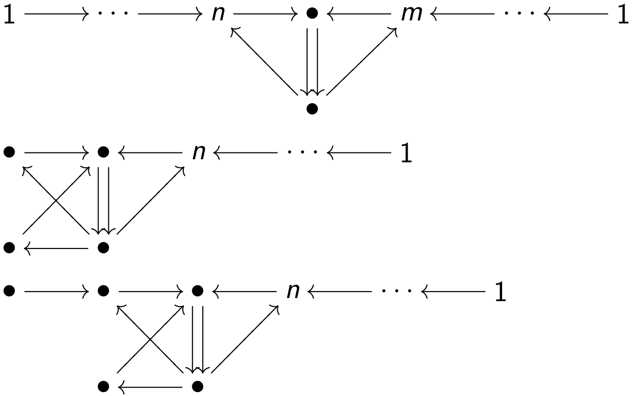
by the Dirac pairing

Alternative (equivalent) forms of the affine quivers

$\widehat{A}(n+1, m+1)$

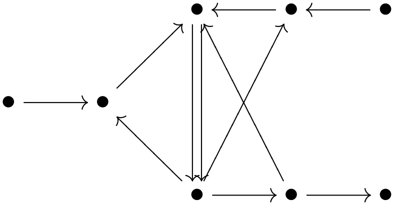
$\widehat{D}_{n+3}$

$\widehat{E}_{n+4}$

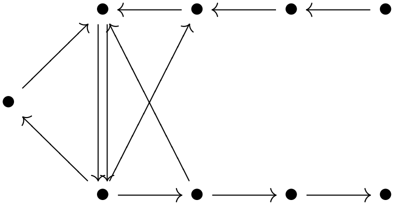


- *SU(2) vector coupled to 0,1,2, or 3 Argyres–Douglas systems*
- *same for elliptic Dynkin quivers*

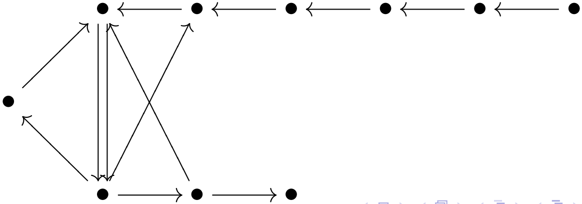
$E_6^{(1,1)}$:



$E_7^{(1,1)}$:



$E_8^{(1,1)}$:



Type IIB engineering of Dynkin $\mathcal{N}=2$ complete theories

Gabrielov: the elliptic Dynkin graphs $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ are the Coxeter–Dynkin graphs of Arnold's parabolic singularities $W(x,y,z)$

$E_6^{(1,1)}$	$x^3+y^3+z^3+\lambda xyz$
$E_7^{(1,1)}$	$x^4+y^4+z^2+\lambda x^2y^2$
$E_8^{(1,1)}$	$x^3+y^6+z^2+\lambda x^2y^2$

$\Rightarrow$ $\mathcal{N}=2$ **model obtained as Type IIB on the local CY**

$$W(x,y,z)+u^2=0$$

- explicit SW curve and differential
- (fractional) monodromy, BPS strong coupling spectrum, *etc.*

see arXiv:1006.3435

CONCLUSIONS

- the algebraic/combinatoric methods are a very powerful tool to study (quiver) $\mathcal{N} = 2$ $4d$ theories
- the special class of *complete* theories is expecially nice and fully classified
- the detailed physics of the generalized Dynkin models *(finite, affine, elliptic)* follows from standard representation theory
- methods and results may be extended to large classes of **non**–complete $\mathcal{N} = 2$: higher rank gauge groups,... `(to appear)`