Counterterms and $E_{7(7)}$ Symmetry in $\mathcal{N} = 8$ Supergravity

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`arxiv:1009.1643`

with H. Elvang, D. Z. Freedman, M. Kiermaier, A. Morales, S. Stieberger

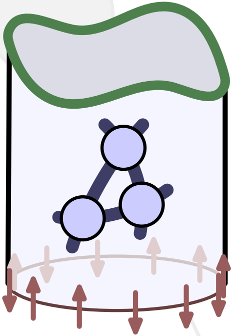
also: `arxiv:0911.5704, 1003.5018, 1007.4813, 1007.5472, 1008.4939, 1009.0743, 1009.1135, 1105.1273, 1105.6087.`

The Simplest QFT

Clearly planar $\mathcal{N}=4$ Super Yang–Mills is the simplest 4D QFT. [Beisert et al. 1012.3982]

We have learned a lot from and about AdS/CFT **integrability**:

- Can apparently solve and compute many physical observables:
  - spectrum of local operators,
  - S-matrix,
  - null polygonal Wilson loops,
  - correlation functions,
  - ... and all other observables?
- There is a lot of **symmetry**:
  - $\mathcal{N}=4$ superconformal symmetry $\mathrm{PSU}(2,2|4)$,
  - T-dual superconformal symmetry $\mathrm{PSU}(2,2|4)$,
  - infinite-dimensional Yangian algebra $\mathrm{Y}(\mathfrak{psu}(2,2|4))$.



Importantly, the model

- is UV finite,
- has a planar limit,
- is related to $AdS_5 \times S^5$ strings via AdS/CFT.

$\mathcal{N} = 8$ Supergravity

$\mathcal{N} = 8$ Supergravity is similar to $\mathcal{N} = 4$ SYM is several respects:

- KLT/BCJ relations: Two copies of (non-planar) $\mathcal{N} = 4$ SYM.
- Low-energy limit of string theory on a flat background.

But there are also important **differences:**

- There is no (tunable) gauge group: no planar limit.
- The symmetries are:
  - local $\mathcal{N} = 8$ supersymmetry,
  - local $\mathrm{SU}(8)$ R-symmetry,
  - global (continuous) $\mathrm{E}_{7(7)}$ electromagnetic duality symmetry.
- Is it UV finite? Do the symmetries rule out counterterms?

This Talk:

- discuss recent results concerning UV finiteness,
- considering invariance under $\mathrm{E}_{7(7)}$,
- using string scattering amplitudes,
- using representation theory of $\mathrm{SU}(2, 2|8)$.

I. Counterterms Anno 2009

Counterterms

Consider $\mathcal{N}=8$ supergravity $S_{\mathcal{N}=8}$ with additional interactions S_k

$$S = S_{\mathcal{N}=8} + \textstyle\sum_k \lambda_k S_k.$$

Question: Which λ_k can receive UV divergences from loops of $S_{\mathcal{N}=8}$?
Assumption: Quantisation around trivial flat supergravity background.

Symmetries:

- manifest symmetries are manifestly respected:
  - Lorentz symmetry $\mathrm{SU}(2,\mathbb{C})$: only scalar counterterms.
  - flavour symmetry $\mathrm{SU}(8)$: only singlet counterterms.
- global translation symmetries are also unbroken:
  - bosonic translations: homogeneous counterterms ($S = \int d^4x\, L$)
  - fermionic translations: supersymmetric counterterms.
    (non-linear action on spacetime fields,
    but linear action on on-shell states for scattering amplitudes)
- electromagnetic duality acts non-linearly:
  - continuous $\mathrm{E}_{7(7)}$ symmetry anomaly-free.

[Bossard Hillmann Nicolai]

First Few Counterterms

Considering Poincaré supersymmetry and $\mathrm{SU}(8)$ flavour symmetry the first few (well-known) counterterm candidates are: [Drummond, Heslop Howe, Kerstan]

$$S = \kappa^{-1} R + \lambda_3 R^4 + \lambda_5 D^4 R^4 + \lambda_6 D^6 R^4 + \lambda_7 D^8 R^4 + \ldots$$

Not made explicit here:

- supersymmetry completion, e.g.: $R^4 \to R^4 + \ldots + D^8 \Phi^4$.
- non-linear completion, e.g.: $R^4 \to R^4 + \Phi^2 R^4 + \ldots$.
- invariance under $\mathrm{E}_{7(7)}$ duality not yet considered.

Dimension counting for loop orders:

tree-level $\sim S_{\mathcal{N}=8} \sim R \sim D^2$, every loop $\sim \kappa^{-1} \sim D^{4-2} = D^2$.

Loop orders for logarithmic UV divergences

	R	R^4	$D^4 R^4$	$D^6 R^4$	$D^8 R^4$	
dimension	2	8	12	14	16	
loops	0	3	5	6	7	$\ldots$
BPS	$\frac{7}{8}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	.	

Other Dimensions

Consider $D^{2k}R^4$ in other spacetime dimensions (dimensional reduction).
Log-divergences at L loops (present/undet./absent) [Bern, Carrasco Dixon, Johansson Kosower, Roiban][Bern, Carrasco Dixon, Johansson, Roiban]

sugra	3	4	5	6	7	8	9	10	11	BPS	
R^4	6	3	2	1.5	1.2	1	0.9	0.8	0.7	$\frac{1}{2}$	
D^4R^4	10	5	3.3	2.5	2	1.7	1.4	1.3	1.1	$\frac{1}{4}$	
D^6R^4	12	6	4	3	2.4	2	1.7	1.5	1.3	$\frac{1}{8}$	
D^8R^4	14	7	4.7	3.5	2.8	2.3	2	1.8	1.6	·	←
$D^{10}R^4$	16	8	5.3	4?	3.2	2.7	2.3	2	1.6	·	
$D^{12}R^4$	18	9	6	4.5	3.6	3	2.6	2.3	2	·	

Compare to UV divergences in maximally supersymmetric Yang–Mills

SYM	3	4	5	6	7	8	9	10	BPS	
F^4	∞	∞	4	2	1.3	1	0.8	0.6	$\frac{1}{2}$	
D^2F^4	∞	∞	6	3	2	1.5	1.2	1	$\frac{1}{4}$	←
D^4F^4	∞	∞	8	4	2.7	2	1.6	1.3	·	
D^6F^4	∞	∞	10	5	3.3	2.5	2	1.7	·	

Critical dimensions/loops: $(d-2)L \geq 14$ (sugra) vs. $(d-4)L \geq 6$ (SYM).

Supergravity Finiteness

Let's be optimistic:

- Does supergravity follow SYM critical dimension $d \geq 4 + 6/L$? [Bern Dixon Roiban]
- Current attempts at 5 loops: $\begin{matrix} D^8 R^4 \\ d \geq 24/5 \end{matrix}$ (sugra) vs. $\begin{matrix} D^{10} R^4 \\ d \geq 26/5 \end{matrix}$ (SYM).

If 5-loop calculation is not appealing, **what other means do we have?**

- Superspace: BPS counterterms $(R^4, D^4 R^4, D^6 R^4)$ are special.
  $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8})$ BPS terms generated only up to $(1, 2, 3)$ loops?
  No divergences at $(3, 5, 6)$ loops?
- First non-BPS counterterm $D^8 R^4$ at 7 loops? [Howe Lindström]

Questions:

- How to rule out R^4, $D^4 R^4$, $D^6 R^4$?
- Can we rule out $D^8 R^4$, too?
- Are there other counterterms? How many? What are they?

Take $\mathrm{E}_{7(7)}$ symmetry into account!

$\mathrm{E}_{7(7)}$ Symmetry of Counterterms

What does $\mathrm{E}_{7(7)}$ symmetry mean?

- Flavour symmetry $\mathrm{SU}(8) \subset \mathrm{E}_{7(7)}$ manifest.
- Coset symmetry $\mathrm{E}_{7(7)}/\mathrm{SU}(8) \simeq \mathbf{70}$ shifts scalars: $\delta\Phi(x) = \Xi + \ldots$.

Now consider **counterterms $D^{2k}R^4$:**

$$D^{2k}R^4 \simeq D^{2k+8}\Phi^4 \simeq D^{2k+4}(D\Phi)^4.$$

- All scalars Φ protected by derivatives $D\Phi$.
- Linearised counterterms $D^{2k}R^4$ invariant under linearized $\mathrm{E}_{7(7)}$.
- Non-linear counterterms may nevertheless violate non-linear $\mathrm{E}_{7(7)}$.
- Claim: Any deformation violates $\mathrm{E}_{7(7)}$ duality symmetry. Finite?! [Kallosh 1104.5480]
- Careful: $\mathrm{E}_{7(7)}$ symmetry requires $\mathrm{E}_{7(7)}$ covariance at $\mathcal{O}(\lambda^2)$. [Bossard Nicolai]

How to address non-linear $\mathrm{E}_{7(7)}$?

- Scattering amplitudes possess non-linear structure.
- Unitarity constructions of S-matrix generate non-linear completion.

II. Scattering Amplitudes

Scattering Amplitudes

Supersymmetry in Scattering Amplitudes:

- Can realise linearly, e.g. spinor helicity $(\lambda, \tilde{\lambda}, \eta)$.
- Implies prefactor $\delta^4(P)\delta^{16}(Q)$ plus 16 constraints from $\bar{Q}$.

$\mathbf{E_{7(7)}}$ Symmetry in Scattering Amplitudes:

- Adler zeros: Amplitude with soft scalars vanishes. [Bianchi Elvang Freedman][Arkani-Hamed Cachazo Kaplan][Kallosh Kugo]
- Double/multiple soft limits (presumably governed by QFT). [Arkani-Hamed Cachazo Kaplan]

S-Matrix Construction:

- Tree level amplitude with counterterms.
- Unitarity: Use recursion relations.
- Or use BCJ relations to lift from $\mathcal{N} = 4$ SYM with counterterms.
- Need to include counterterms in recursion relations. Tough!

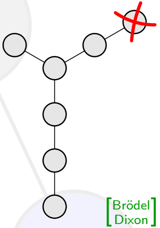
String Scattering Amplitudes:

- Effective action with active couplings $\lambda_3, \lambda_5, \lambda_6, \lambda_7, \ldots$.
- Calculate scattering amplitudes in string theory.

Open/Closed String Scattering

Supergravity Amplitudes from string theory:

- Compute some open string scattering.
- Use supersymmetry to recover missing components. [Stieberger Taylor] [Stieberger Taylor]
- KLT relations predict closed string amplitudes.
- Torus compactification 10D $\to$ 4D + 6d.
- Coset for compactified string: $\mathrm{SO}(6,6)/\mathrm{SO}(6)\times\mathrm{SO}(6)$.
- R-symmetry broken $\mathrm{SU}(8)\to\mathrm{SU}(4)\times\mathrm{SU}(4)\times\mathrm{U}(1)$.



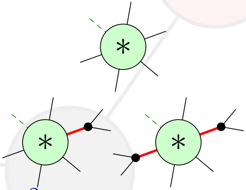
$\mathrm{E}_{7(7)}$ Symmetry?

- Consider 6 vertex operator insertions $\langle ++--\Phi\Phi\rangle$. [Brödel Dixon]
- Average over $\mathrm{SU}(8)$ to recover supergravity $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ coset. [Elvang Kiermaier]
- Double soft scalar limit respected. [Brödel Dixon]
- Single soft scalar limit violated. [Brödel Dixon] [Elvang Kiermaier] [NB, Elvang Freedman, Kiermaier Morales, Stieberger]

Finiteness:

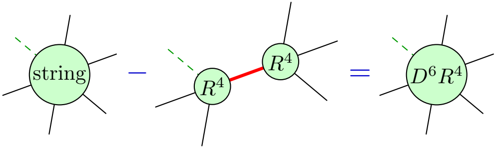
- 3,5,6-loop counterterms R^4, D^4R^4, D^6R^4 excluded.

$\mathrm{E}_{7(7)}$-Violation of BPS Counterterms



How is $\mathrm{E}_{7(7)}$ violated quantitatively?

- Leading-leg single soft limit is local (polynomial).
  Higher-leg soft limits are rational.
- $\mathrm{E}_{7(7)}$-Violation originates from counterterms.
- Non-linear completion of $\begin{cases} R^4 \to R^4(1 - \frac{3}{10}\Phi^2 + \ldots), \\ D^4R^4 \to D^4R^4(1 - \frac{3}{14}\Phi^2 + \ldots), \\ D^6R^4 \to D^6R^4(1 - \frac{3}{14}\Phi^2 + \ldots). \end{cases}$
- Counterterm D^6R^4 requires special attention. [NB, Elvang Freedman, Kiermaier Morales, Stieberger]
  Competing contributions from $\lambda_6 \sim \zeta(3)^2$ and $(\lambda_3)^2 \sim \zeta(3)^2$.
  Subtract contribution from R^4—R^4 for $\mathrm{E}_{7(7)}$-violation of D^6R^4.



Automorphism Properties on BPS Counterterms

Independent check using automorphic functions. [Bossard Howe Stelle][NB, Elvang Freedman, Kiermaier Morales, Stieberger]

Suppose non-linear completion has scalar dependence

$$f_3(\Phi)R^4, \qquad f_5(\Phi)D^4R^4, \qquad f_6(\Phi)D^6R^4.$$

The functions f_k must satisfy the Laplace equations [Green Russo Vanhove][Green Russo Vanhove][Green, Miller Russo, Vanhove]

$$(\Delta+42)f_3=0, \qquad f_3 \sim \lambda_3(1-\tfrac{3}{10}\Phi^2+\ldots),$$
$$(\Delta+30)f_5=0, \qquad f_5 \sim \lambda_5(1-\tfrac{3}{14}\Phi^2+\ldots),$$
$$(\Delta+30)f_6=-(f_3)^2, \qquad f_6 \sim \lambda_6(1-\tfrac{3}{14}\Phi^2+\ldots).$$

- Coefficents match with scattering amplitude analysis. [NB, Elvang Freedman, Kiermaier Morales, Stieberger]
- Contribution $-(f_3)^2$ corresponds to R^4—R^4 term.

III. 7-Loop Counterterms

7-Loop Counterterm D^8R^4

What about 7-loop counterterms?

- Many 7-loop counterterms exist D^8R^4, D^4R^6, ...
- One counterterm D^8R^4 at 4 legs.
- Two counterterms D^4R^6 at 6 legs. ...

$\mathrm{E}_{7(7)}$ Symmetry? [NB, Elvang Freedman, Kiermaier Morales, Stieberger]

- Straight string calculation violates $\mathrm{E}_{7(7)}$ single soft limit at 6 legs. No contribution from composite R^4, D^4R^4, D^6R^4 terms.
- Both D^4R^6 counterterms violate $\mathrm{E}_{7(7)}$ single soft limit at 6 legs.
- Can add specific D^4R^6 combination to restore $\mathrm{E}_{7(7)}$. Non-linear completion of D^8R^4 respects $\mathrm{E}_{7(7)}$ at 6 points.

Questions?

- Does non-linear D^8R^4 violate $\mathrm{E}_{7(7)}$ at higher points?
- How many supersymmetric counterterms at 7 loops?
- How many preserve $\mathrm{E}_{7(7)}$ symmetry?
- Is D^8R^4 the superspace volume $\int d\theta^{32} e$?

Descendants

7-loop counterterms are full superspace integrals: [Kallosh 1009.1135][NB, Elvang Freedman, Kiermaier Morales, Stieberger]

$$D^8 R^4 \simeq D^{16}\Phi^4 \simeq Q^{32}\Phi^4 \simeq \int d^{32}\theta\,\Phi^4,$$
$$D^4 R^6 \simeq D^{16}\Phi^6 \simeq Q^{32}\Phi^6 \simeq \int d^{32}\theta\,\Phi^6,$$
$$\ldots$$

Scalar descendants $Q^{32}\Phi^n$:

- Supersymmetric by construction.
- Non-vanishing for $n \geq 4$.
- One counterterm for every $\mathrm{SU}(8)$ singlet in $\mathbf{70}^{\otimes_s n}$ (n even).

n	4	6	8	10	12	14	16	$\ldots$
#	1	2	3	4	6	8	10	$\ldots$

- Infinitely many counterterms.
- All 7-loop counterterms of this form.

$\mathrm{E}_{7(7)}$-Invariance

Single soft limit of an amplitude transforms in $\mathbf{70}$. [Kallosh 1009.1135][NB, Elvang Freedman, Kiermaier Morales, Stieberger]

Operators:

n	4	6	8	10	12	14	16
$\mathbf{1}$	1	2	3	4	6	8	10
$\mathbf{70}$	0	2	4	6	9	14	19

- All soft limits of Φ^n non-vanishing.
- Soft limits of $Q^{32}\Phi^n$ non-vanishing for $n > 4$: $Q^{32}\Phi^n$ violates $\mathrm{E}_{7(7)}$.
- Soft limit of $Q^{32}\Phi^4$ vanishes (BPS): $Q^{32}\Phi^4$ respects $\mathrm{E}_{7(7)}$.

Non-linear completion of $D^8R^4 \simeq Q^{32}\Phi^4$?

- **6 legs:** Soft limit in 2d space of $\mathbf{70}$ operators.
  2 $\mathrm{E}_{7(7)}$-violating counterterms to adjust.
- **8 legs:** Soft limit in 4d space. Only 3 counterterms to adjust.
  $\mathrm{E}_{7(7)}$ violation at 8 legs?

Cohomology

Single soft limit acts like constant shift of scalars $\partial/\partial\Phi$.

- Exterior derivative d on space of scalars $\mathbb{R}^{70}$. [Bossard unpublished]
- de Rham complex: $\mathbf{1}$, $\mathbf{70}$, $\mathbf{70}^{\otimes_a 2} = \mathbf{2352} \oplus \mathbf{63}$, $\ldots$. [NB unpublished]

n	4	6	8	10	12	14	16
1	1	2	3	4	6	8	10
70	·	2	4	6	9	14	19
2352	·	·	1	2	3	6	10
…	·	·	·				

Single soft limit dA is closed one-form ($d^2 A = 0$). **8 legs:**

- Two-form $f_{8,2}$ is exact $f_{8,2} = df_{8,1}$. 1/4 one-forms are not closed.
- Remaining 3/4 closed one-forms are exact $f_{8,1} = df_{8,0}$.
- $\mathrm{E}_{7(7)}$-violation can be repaired by non-linear completion $dA = df_{8,0}$.

Higher legs:

- de Rham cohomology on $\mathbb{R}^{70}$ is trivial. No obstructions!
- Non-linear $\mathrm{E}_{7(7)}$-invariant counterterm $D^8 R^4 + \ldots$ exists. [Bossard unpublished]

Superspace Volume

Superspace volume $\int d\theta^{32} e$:

- is a supersymmetric singlet,
- is a 7-loop counterterm,
- is manifestly $\mathrm{E}_{7(7)}$-invariant.

Questions:

- Does it vanish?
- Otherwise, does it match the non-linear $D^8 R^4$?

Curious recent result using harmonic superspace: [Bossard, Howe Stelle, Vanhove]

- Superspace volume vanishes!
- There is an alternative $\mathrm{E}_{7(7)}$-invariant counterterm.
- Counterterm not a full superspace integral.

Implications:

- Still 7-loop UV divergence to be expected …
- … unless BPS counterterm ruled out (by some unknown mechanism)?

IV. Counterterm Census

Counterterm Enumeration

Let us **enumerate counterterms** using superconformal symmetry:

- Free supergravity fields form $\mathrm{SU}(2,2|8)$ superconformal multiplet.
- Supersymmetric operators at top of superconformal multiplets. [Drummond, Heslop, Howe]

Procedure: [NB, Bianchi, Morales, Samtleben] [NB, Elvang, Freedman, Kiermaier, Morales, Stieberger]

- Generate field multiplet (scalars $D^k\Phi$, ..., Weyl tensor D^kR)

$$F(x,\ldots)=\sum_{k=0}^{\infty}\Big([\mathbf{70},k,k]x^k-\ldots+[\mathbf{1},k+4,k]x^{k+2}\Big)$$

- Enumerate all graded symmetric products (up to some dimension).
- Decompose into irreps of $\mathrm{SU}(2,2|8)$.
- One counterterm for each irrep with singlet top component.
  Benefit: consider bottom rather then top components, Q^{32} for free.

List of Counterterms

n-pt $N^{(n+k)/2}MHV$ at L Loops

n = 4, k = 0

L	3	4	5	6	7	8	9	10	11	12	13
blue	1	0	1	1	1	1	2	1	2	2	2

n = 5, k = 1

L	8	9	10	11	12
blue	1	1	3	4	7

n = 6, k = 0

L	3	4	5	6	7	8	9	10	11	12
blue					2	3	12	33	90	196
red	1	0	1	1	2	3	5	6	10	12

n = 6, k = 2

L	9	10	11	12
blue	2	2	7	15

n = 7, k = 1

L	8	9	10	11	12
blue	3	14	90	360	
red	4	8	28	65	145

n = 7, k = 3

L	10	11
blue	1	2

n = 8, k = 0

L	7	8	9	10	11	12
blue	3	8	117	865	5209	
red	4	17	122	553	2062	
green	1	9	24	71	163	350

n = 8, k = 2

L	9	10	11
blue	7	48	397
red	5	20	102

n = 8, k = 4

L	11
blue	2

n = 9, k = 1

L	8	9	10	11
blue	8	123	1832	
red	16	194	1747	10 292
green	9	77	404	1582

n = 9, k = 3

L	10	11
blue	11	
red	6	36

blue: counterterm, red: single soft limit, green: two-form

Enumerated all local operators up to $2L < 30 - n$: [NB unpublished]

- 4.8×10^{22} local operators (3.5 hours),
- 8.8×10^{5} multiplets with multiplicity (42 hours).

Cohomology

Cohomology of single soft limit d on supersymmetric singlets:

- H^0: counterterms invariant under linear $\mathrm{E}_{7(7)}$,
- H^1: non-linear $\mathrm{E}_{7(7)}$-violations.

3–6 loops: BPS counterterms.

L	3	4	5	6
H^0 at 4 legs	1	0	1	1
H^1 at 6 legs	1	0	1	1

Non-linear counterterms R^4, D^4R^4, D^6R^4 break $\mathrm{E}_{7(7)}$.

7 loops:

- $H^0 = \langle D^8R^4 \rangle = \mathbb{R}^1$,
- $H^1 = \{0\} = \mathbb{R}^0$.
- 1 non-linear invariant.

8 loops:

- $H^0 = \langle D^{10}R^4, \mathrm{Re}\, D^8R^5, \mathrm{Im}\, D^8R^5 \rangle = \mathbb{R}^3$,
- $H^1 = \langle \mathrm{Re}\, D^6R^6 d\Phi, \mathrm{Im}\, D^6R^6 d\Phi \rangle = \mathbb{R}^2$.
- 1 non-linear invariant (?)

Problem: Guesswork! Plain counting usually not sufficient.

V. Conclusions

Conclusions

Investigated $\mathcal{N} = 8$ Supergravity Counterterms by

- $\mathrm{SU}(8)$ averaged string scattering amplitudes,
- $\mathrm{SU}(2,2|8)$ representation theory,
- cohomology.

Results: Supersymmetric counterterms

- 3 BPS counterterms at 3,5,6 loops.
- Infinitely many non-BPS counterterms for every $L \geq 7$.

$\mathbf{E_{7(7)}}$ **symmetry** constraints:

- All BPS counterterms excluded. Finitely many at every $L \geq 7$.
- $\mathcal{N} = 8$ supergravity finite at $L < 7$! One counterterm at $L = 7$.

Outlook:

- Is there a larger symmetry (or other mechanism) to rule out $D^8 R^4$?
- Is $\mathcal{N} = 8$ supergravity finite at $L \geq 7$? Why?!

Questions upon questions ...